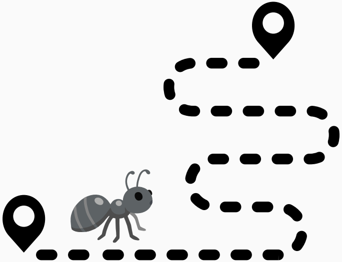
A probabilistic reinforcement-learning algorithm to find shortest paths in a graph

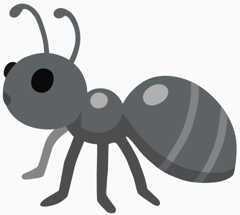
Zoé Varin

June 10th, 2025

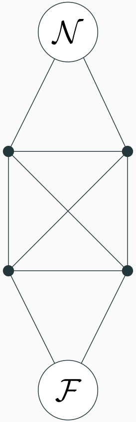
Joint work with Cécile Mailler



Introduction



Definition of the model (one-nest version)



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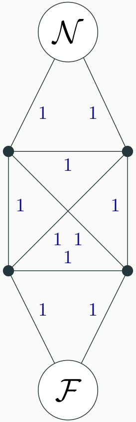

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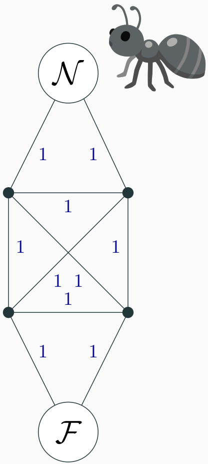

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•**weighted random walk**:

$$\mathbb{P}\left(u \to v\right) = \frac{W_{uv}(n)}{\sum_{e:u\in e} W_e(n)}$$

stopped at $\mathcal{F}$

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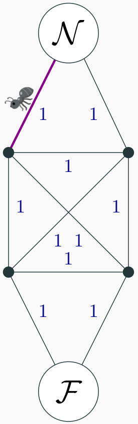

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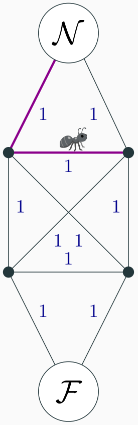

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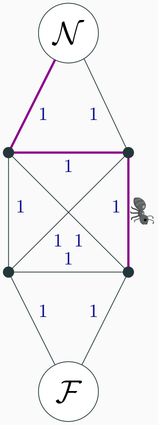

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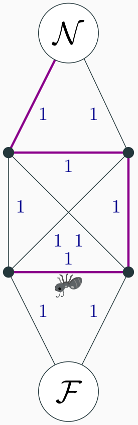

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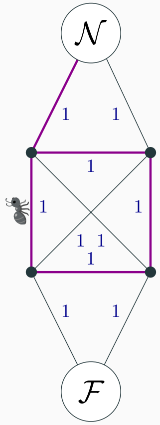

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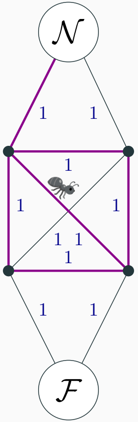

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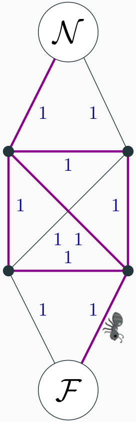

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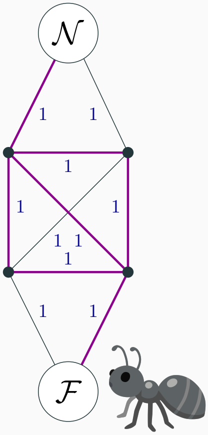

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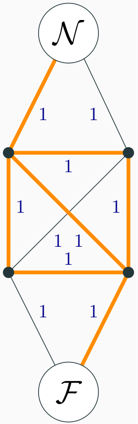

(T) trace

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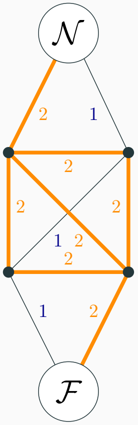

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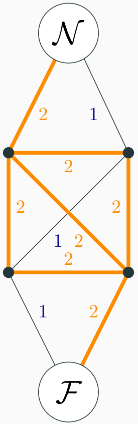

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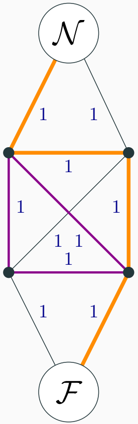

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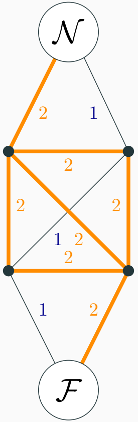

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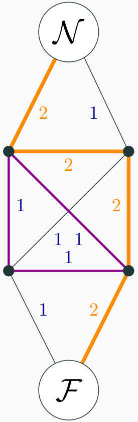

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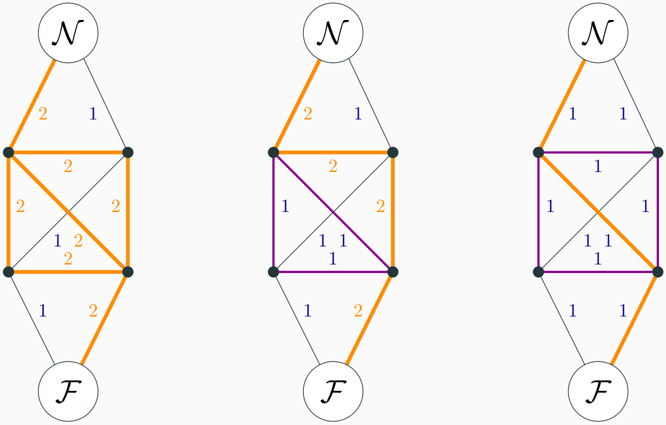

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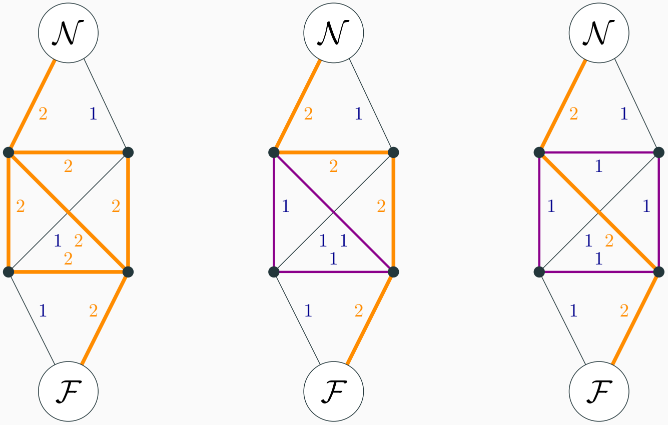

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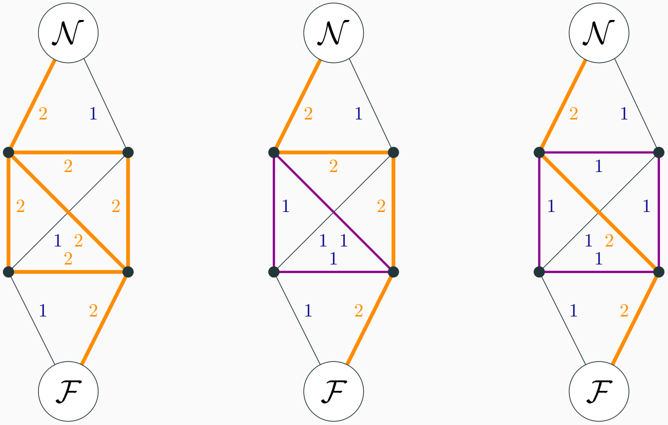

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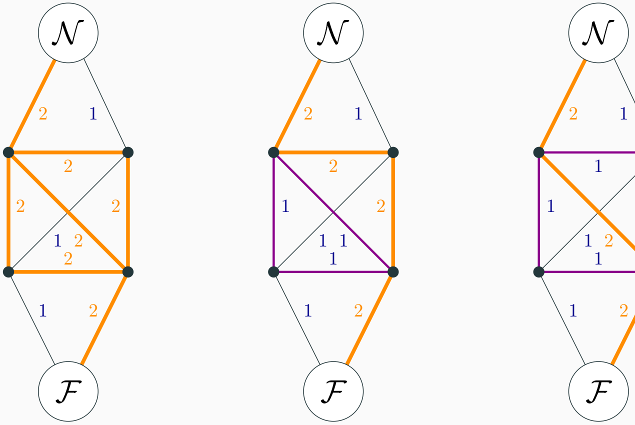

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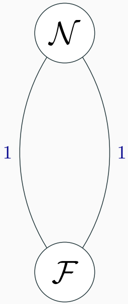
Question: Do the ants find shortest paths from $\mathcal{N}$ to $\mathcal{F}$?

A warm-up and a Pólya urn



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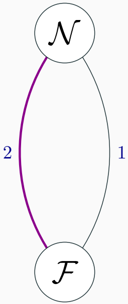

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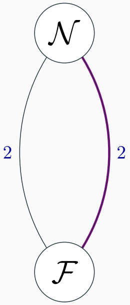

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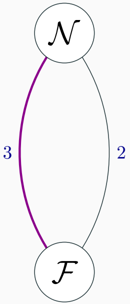

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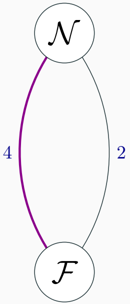

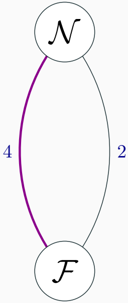

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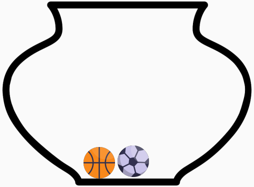

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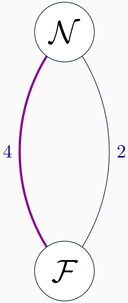

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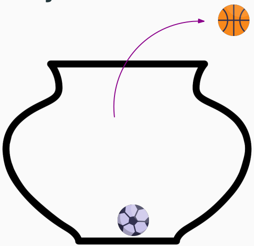

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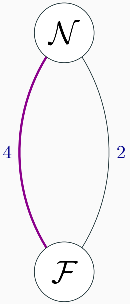

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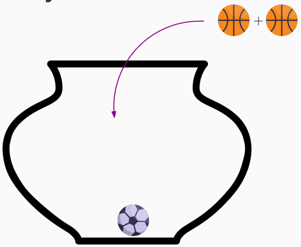

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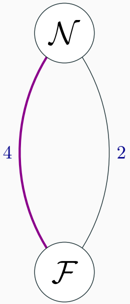

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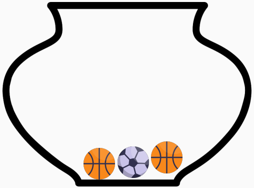

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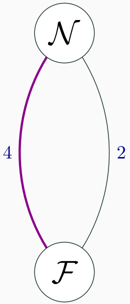

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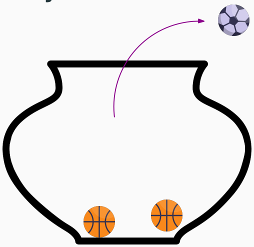

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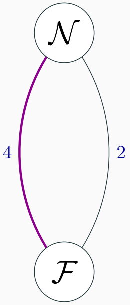

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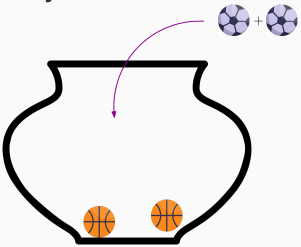

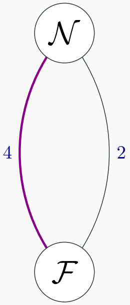

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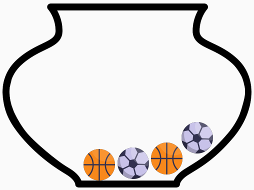

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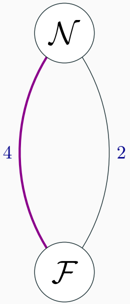

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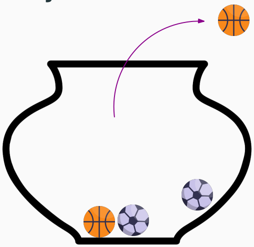

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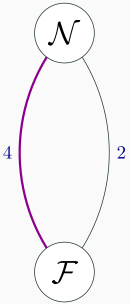

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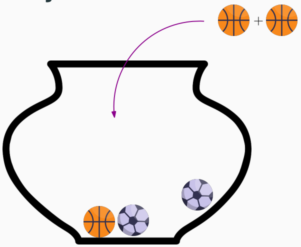

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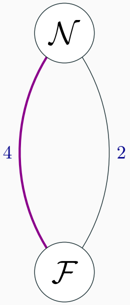

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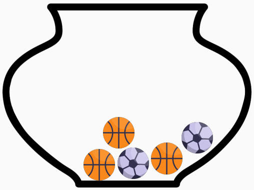

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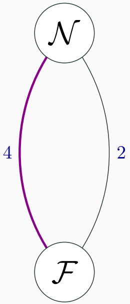

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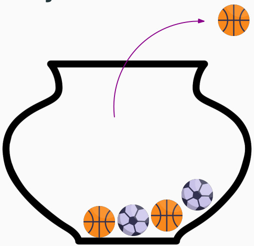

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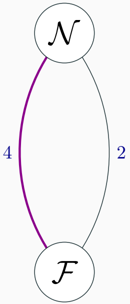

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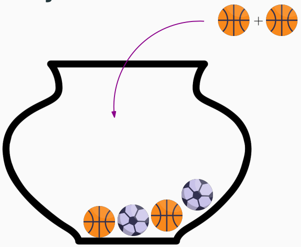

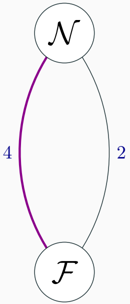

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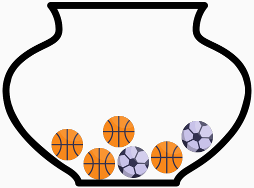

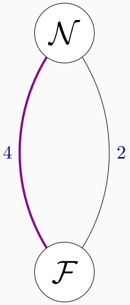

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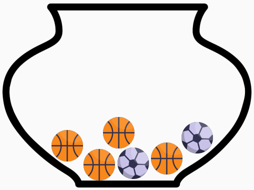


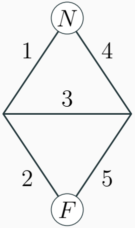
Asymptotic behavior:

Almost surely,

$$\frac{\#🏀}{n} \xrightarrow[n\to\infty]{} U \sim \mathcal{U}([0,1])$$

Geodesic (G) model on the lozenge graph

The lozenge graph:



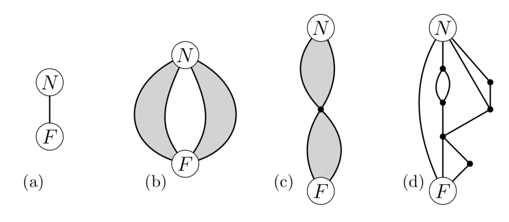
Theorem (Kious, Mailler, Schapira [KMS22a])

Almost surely,

$$\frac{W_i(n)}{n} \xrightarrow[n\to\infty]{} \chi_i, \quad \forall 1 \leq i \leq 5$$

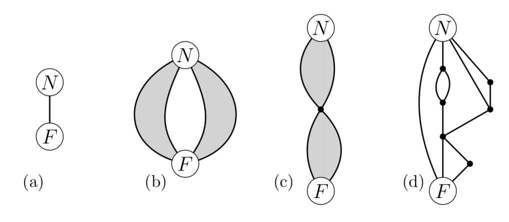
where $(\chi_i)_{1\leq i\leq 5}$ is a random vector, such that almost surely, $\chi_1 = \chi_2 = 1 - \chi_4 = 1 - \chi_5 \in (0,1)$ and $\chi_3 = 0$.

Recursive definition of series-parallel (SP) graphs (image from [KMS22a]):



Loop-erased (LE) model on series-parallel graphs

Recursive definition of series-parallel (SP) graphs (image from [KMS22a]):



Theorem (Kious, Mailler, Schapira [KMS22a])

If G is a SP graph, then in the loop-erased (LE) model, almost surely,

$$\frac{W_e(n)}{n} \xrightarrow[n\to\infty]{} \chi_e, \quad \forall e \in E$$

where $(\chi_e)_{e\in E}$ is a random vector such that $\forall e$, $\chi_e \neq 0$ if and only if e belongs to a shortest path from N to F.

Conjecture for the loop-erased (LE) and geodesic (G) models

Conjecture [KMS22a]

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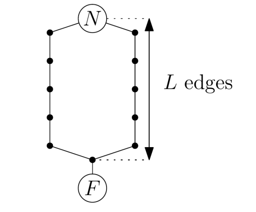
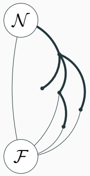


For L large enough, there exists e such that

$$\mathbb{P}\left(W_e(n)/n \to 0\right) > 0$$

Trace (T) model ([KMS22b])

G is *tree-like* if $G \backslash \{\mathcal{F}\}$ is a tree.



Theorem [KMS22b]

If $G = (V, E)$ is *tree-like* and $a = \{\mathcal{N}, \mathcal{F}\} \in E$ with multiplicity 1, then

$$\frac{W_a(n)}{n} \to 1 \quad \text{and} \quad \frac{W_e(n)}{n} \to 0, \quad \forall e \in E \backslash \{a\}$$

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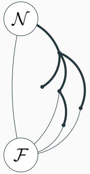

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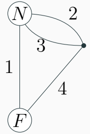
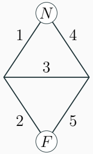
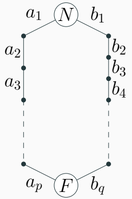
Other examples: the cone, the lozenge and the (p, q)-path



$$\frac{W(n)}{n} \xrightarrow[n\to\infty]{} (1, 1/3, 1/3, 0)$$



$$\frac{W(n)}{n} \xrightarrow[n\to\infty]{} (w^\star, 1/2, 1/2, w^\star, 1/2)$$

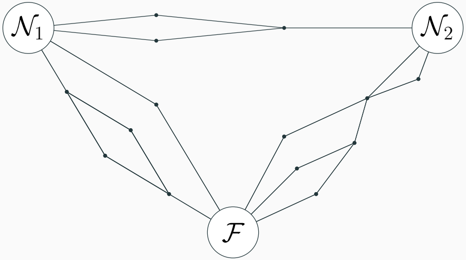


$$\frac{W_{a_k}(n)}{n} \xrightarrow[n\to\infty]{} \alpha^k, \quad \frac{W_{b_k}(n)}{n} \xrightarrow[n\to\infty]{} \beta^k$$

Conjecture: deterministic limit for any graph without multiple-edges adjacent to F.

Multinest version

Multinest-version: at every step n, $\mathcal{N}(n) = \begin{cases} \mathcal{N}_1 & \text{with proba } \alpha_1 \in (0,1) \\ \mathcal{N}_2 & \text{with proba } \alpha_2 = 1 - \alpha_1 \end{cases}$.



Multinest version on triangle-SP graphs

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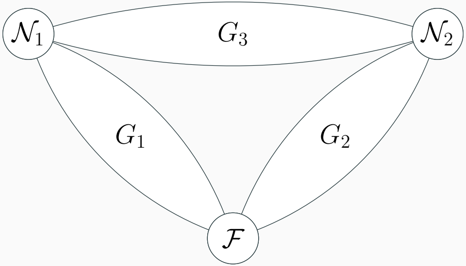
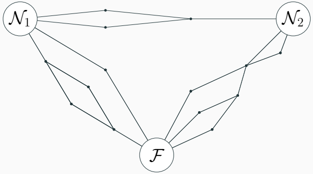


Triangle-SP graph: G_1, G_2, G_3 series-parallel graphs

Our main result: the loop-erased (LE) model on triangle-SP graphs



For $i \in \{1, 2, 3\}$,

- $\ell_i := h_{\min}(G_i)$ distance between the source and the sink of G_i.
- $N_i(n)$ number of reinforcement in G_i before step n.

Remark: $\forall n, N_1(n) + N_2(n) = n$.

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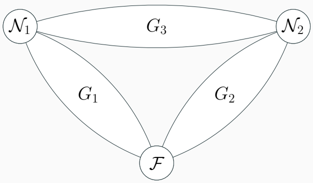

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Theorem (Mailler, V. 2025+)

We assume that $\ell_1 \leq \ell_2$. Almost surely,

- *if $\ell_2 \geq \ell_1 + \ell_3$, then $\frac{N_1(n)}{n} \underset{n\to\infty}{\longrightarrow} 1$ and $\frac{N_3(n)}{n} \underset{n\to\infty}{\longrightarrow} \alpha_2$,*

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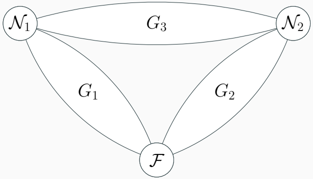

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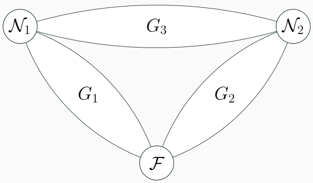

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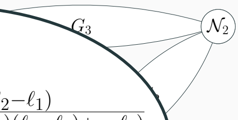

$$\beta_1 = \frac{\alpha_1\ell_1(\ell_3+\ell_2-\ell_1)}{\ell_1\ell_3+(\ell_2-\ell_1)((1-\alpha_1)(\ell_3-\ell_2)+\alpha_1\ell_1)}$$

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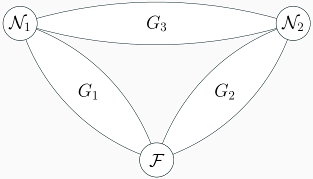

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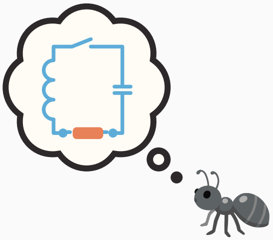
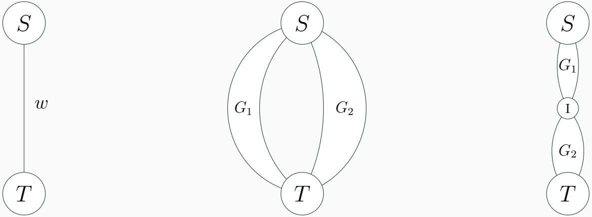
Moreover, almost surely: for all $e \in G_i$, $\frac{W_e(n)}{n} \underset{n\to\infty}{\longrightarrow} \xi_e$, with $\xi_e \neq 0$ if and only if $\lim N_i(n)/n > 0$ and e belongs to a shortest path between two vertices of $\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{F}\}$.

Toolbox & Proof

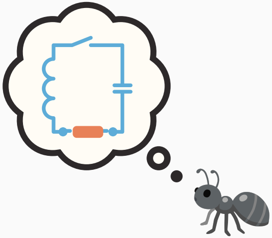
Conductance method



Effective conductance between two vertices - **recursive definition:**



(a) $C_G = w$ **(b)** $C_G = C_{G_1} + C_{G_2}$ **(c)** $C_G = \frac{1}{1/C_{G_1} + 1/C_{G_2}}$



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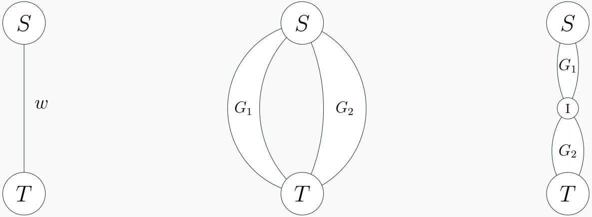

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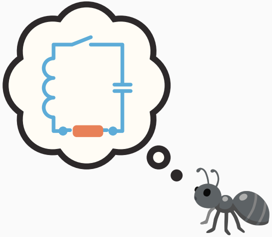
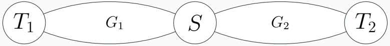


Key idea: the probability that a random walk starting from S hits T_1 before T_2 is $\frac{C_{G_1}}{C_{G_1}+C_{G_2}}$.



On SP graphs:

Theorem (Kious, Mailler, Schapira [KMS22a])

$$\frac{n}{h_{\max}(G)} \le C_G(n) \le \frac{n+C}{h_{\min}(G)}$$

Conductances in the (LE) model on SP graphs

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Theorem (Kious, Mailler, Schapira [KMS22a])

There exists a random variable K and constants α, C such that

$$\frac{n - Kn^{\alpha}}{h_{\min}(G)} \leq C_G(n) \leq \frac{n + C}{h_{\min}(G)}$$

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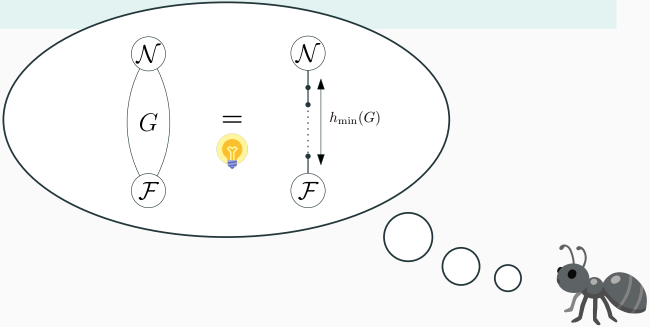
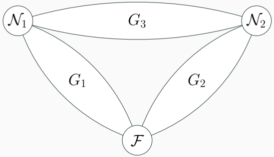


Preliminary computation



Example: If $\mathcal{N}(n) = \mathcal{N}_1$, the probability to reinforce in G_1 is

$$\frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}}$$

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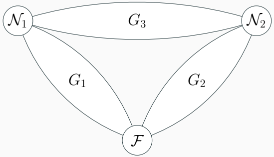

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Key to apply [KMS22a] results:

- conditionnal on $\gamma \in G_1$, γ is distributed as γ_1 obtained by doing a (LE) step in G_1 only
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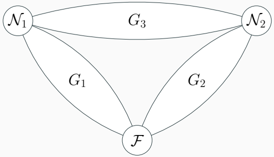

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Corollary

For every $i \in \{1, 2, 3\}$,

$$\frac{C_{G_i}(n)}{N_i(n)} \underset{n\to\infty}{\longrightarrow} \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

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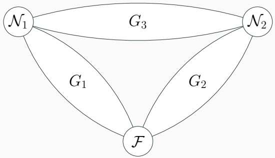

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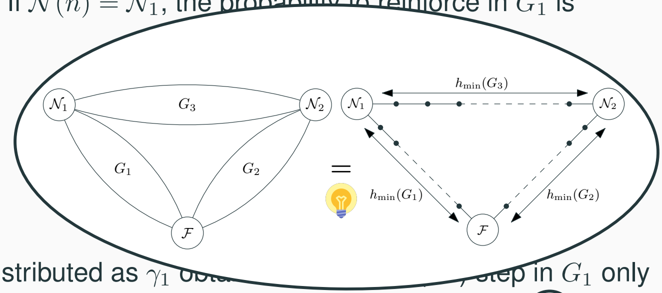

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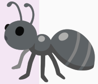

Stochastic approximation

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} = X_n + \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

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- the noise ξ_{n+1} is $\mathcal{F}_{n+1}$-measurable and such that $\forall n, \mathbb{E}_n\left[\xi_{n+1}\right] = 0$,
- the remainder term r_n is $\mathcal{F}_n$-measurable and such that $\sum_n n^{-1}||r_n|| < \infty$ a.s.

Stochastic approximation

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} = X_n + \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

and if

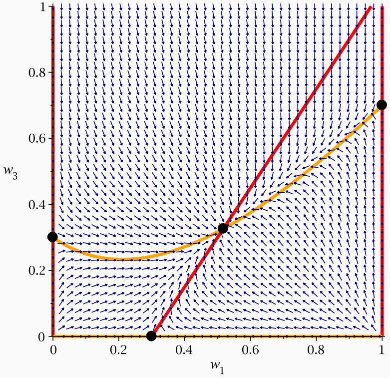
- $(X_n)_{n\geq 0}$ is adapted to some filtration $(\mathcal{F}_n)_{n\geq 0}$, and takes value in some convex compact $\mathcal{E} \subseteq \mathbb{R}^d$,
- $F : \mathcal{E} \to \mathbb{R}^d$ is a Lipschitz function,
- the noise ξ_{n+1} is $\mathcal{F}_{n+1}$-measurable and such that $\forall n, \mathbb{E}_n \left[\xi_{n+1}\right] = 0$,
- the remainder term r_n is $\mathcal{F}_n$-measurable and such that $\sum_n n^{-1}||r_n|| < \infty$ a.s.

Claim: the process $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n\geq 0}$ is a stochastic approximation !

Illustration of the ODE method

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} - X_n = \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$



ODE method

If there exists $p_1, \dots, p_k$ s.t. for any $w \in [0,1]^2$, the solution of the ODE $\dot{y} = F(y)$ starting at w converges to some p_i, then almost surely,

$$\exists i : \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \xrightarrow[n\to\infty]{} p_i$$

Main idea: if ξ_{n+1} and r_n behave nicely, $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ *follows the flow of the ODE* $\dot{y} = F(y)$!

Prove that our process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right)$$

$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

Prove that our process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

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$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$

Prove that our process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\left(I\right.$$

For every $i \in \{1,2,3\}$,

$$\frac{C_{G_i}(n)}{N_i(n)} \longrightarrow \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$

Prove that our process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\Big(I$$

> For every $i \in \{1, 2, 3\}$,
>
> $$\frac{C_{G_i}(n)}{N_i(n)} \longrightarrow \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$

$$\sim \alpha_1 \frac{w_1/\ell_1}{w_1/\ell_1 + \frac{w_2/\ell_2 \, w_3/\ell_3}{w_2/\ell_2 + w_3/\ell_3}} + \alpha_2\left(1 - \frac{w_2/\ell_2}{w_2/\ell_2 + \frac{w_1/\ell_1 \, w_3/\ell_3}{w_1/\ell_1 + w_3/\ell_3}}\right) =: p_1(w_1, w_2, w_3) \text{ (with } w_i = \hat{N}_i(n))$$

Prove that our process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right)$$

$$= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - p(\hat{N}(n)) + p(\hat{N}(n)) - \frac{N(n)}{n}\right)$$

$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$

$$\sim \alpha_1 \frac{w_1/\ell_1}{w_1/\ell_1 + \frac{w_2/\ell_2 \, w_3/\ell_3}{w_2/\ell_2 + w_3/\ell_3}} + \alpha_2\left(1 - \frac{w_2/\ell_2}{w_2/\ell_2 + \frac{w_1/\ell_1 \, w_3/\ell_3}{w_1/\ell_1 + w_3/\ell_3}}\right) =: p_1(w_1, w_2, w_3) \text{ (with } w_i = \hat{N}_i(n))$$

Prove that our process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

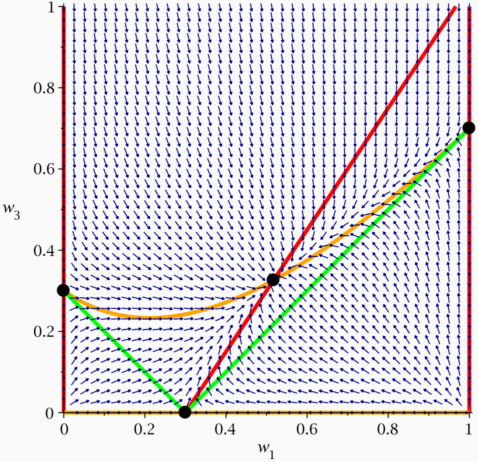
$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right)$$
$$= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - p(\hat{N}(n)) + p(\hat{N}(n)) - \frac{N(n)}{n}\right)$$
$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

And $\sum_n \frac{||r_n||}{n} < \infty$, because $\forall i \in \{1,2,3\}, N_i(n) \geq n^{\varepsilon_i}$.

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$
$$\sim \alpha_1 \frac{w_1/\ell_1}{w_1/\ell_1 + \frac{w_2/\ell_2 w_3/\ell_3}{w_2/\ell_2+w_3/\ell_3}} + \alpha_2\left(1 - \frac{w_2/\ell_2}{w_2/\ell_2 + \frac{w_1/\ell_1 w_3/\ell_3}{w_1/\ell_1+w_3/\ell_3}}\right) =: p_1(w_1, w_2, w_3) \text{ (with } w_i = \hat{N}_i(n))$$

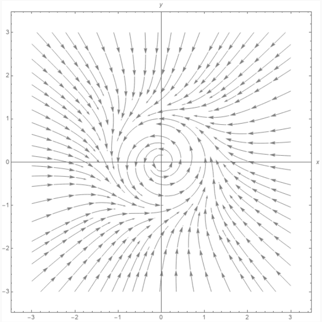
Convergence of the process thanks to the ODE method

Vector field: $F(w_1, w_3)$



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

What does not happen:

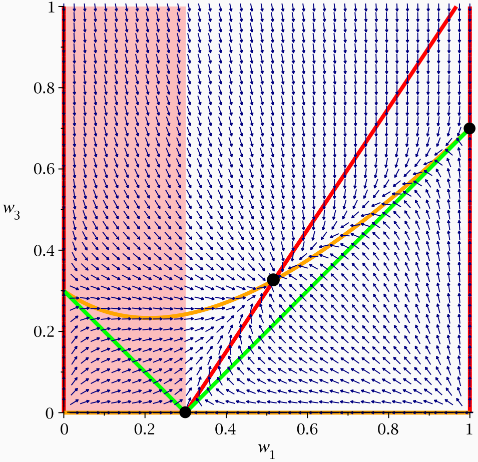


Conclusion: any solution to $\dot{y} = F(y)$ starting in $[0,1]^2$ converges

$\rightarrow \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ converges !

Eliminating the “bad” zeros

Vector field: $F(w_1, w_3)$



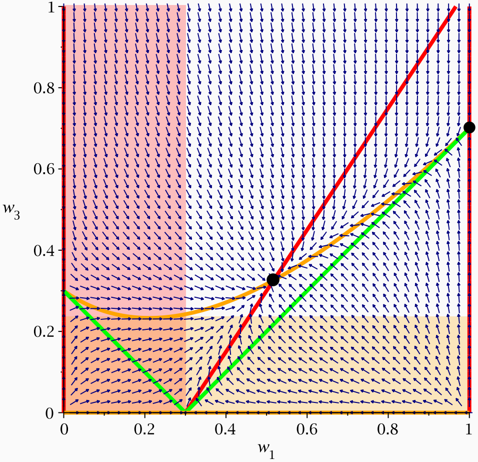
(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$

Eliminating the “bad” zeros

Vector field: $F(w_1, w_3)$



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

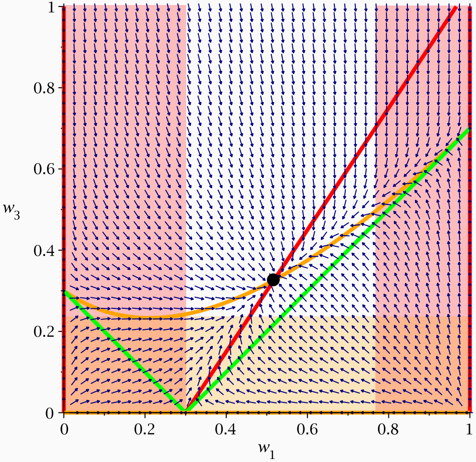
Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

Eliminating the "bad" zeros

Vector field: $F(w_1, w_3)$



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

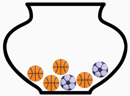
- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Urn models

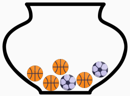
$N(n) := \#$🏀 at step n. In a classical Pólya urn:

$$\mathbb{P}\left(N(n+1) = N(n) + 1 \,\middle|\, \frac{N(n)}{n} = w\right) = \quad w$$

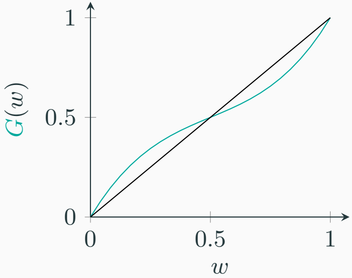


Urn models

$N(n) := \#$🏀 at step n. In a G-urn:

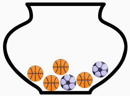


$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$



Urn models

$N(n) := \#$🏀 at step n. In a G-urn:

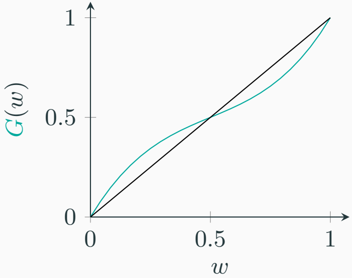


$$\mathbb{P}\left(N(n+1) = N(n) + 1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

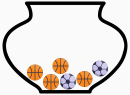
Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} W$, where W is a (random) stable fixed point of G.



$N(n) :=$ #  at step n. In a G-urn:

$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$



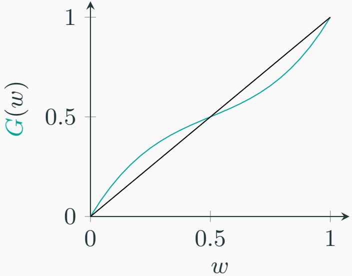
w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \xrightarrow[n\to\infty]{} W$, where W is a (random) stable fixed point of G.

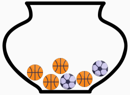
Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0,1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$, $W = 0.5$ a.s.



Urn models

$N(n) := \#$🏀 at step n. In a G-urn:



$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \xrightarrow[n\to\infty]{} W$, where W is a (random) stable fixed point of G.

Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0,1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$, $W = 0.5$ a.s.

💡 Use this on our two-dimensional process $(N_1(n), N_3(n))$

If, for any $x_3 \in [0,1]$,

$$\underbrace{\mathbb{P}\left(N_1(n+1) = N_1(n)+1 \,\middle|\, \frac{N_1(n)}{n} = w_1, \frac{N_3(n)}{n} = w_3\right)}_{\sim F(w_1,w_3)} \geq G(w_1)$$

and if every stable fixed point of G is larger than some c, then

$$\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq c$$

Conclusion in the different cases

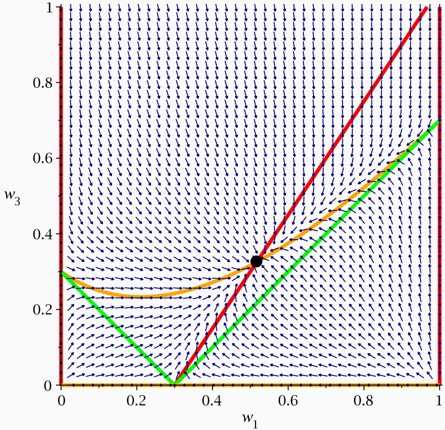


Figure 3: $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$. $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (\beta_1, \beta_3)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Conclusion in the different cases

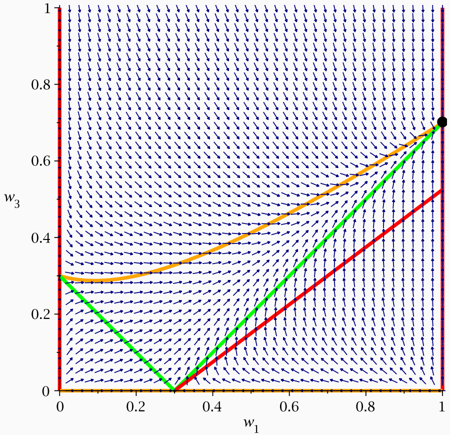


Figure 3: $\ell_1 = 2$, $\ell_2 = 6$ and $\ell_3 = 3$. $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (1, \alpha_2)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

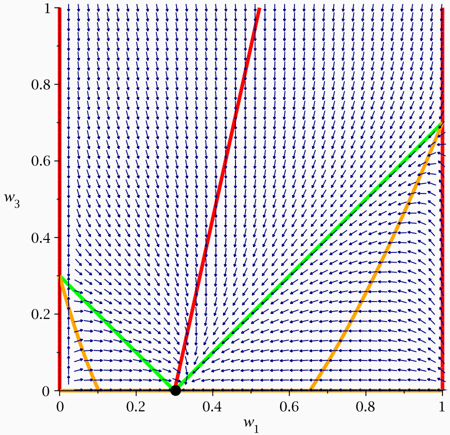


Figure 3: $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 9$. $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (\alpha_1, 0)$

Lemma

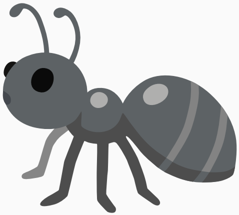
- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Thank you !



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