Continuous (and discrete) dispersion models on the circle

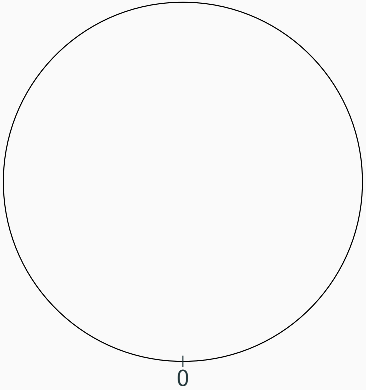
Zoé Varin

November 27th, 2025

Joint work with Jean-François Marckert

Definition of the model

State space $\mathcal{C} = \mathbb{R}/\mathbb{Z}$. $m_0, \cdots, m_n$ with $\sum m_i < 1$.



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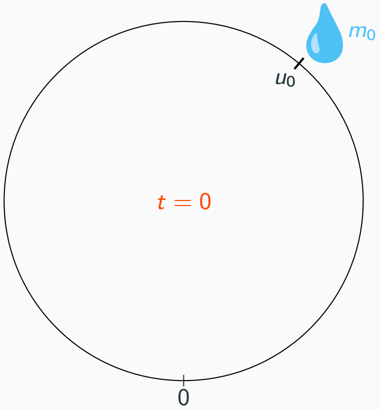

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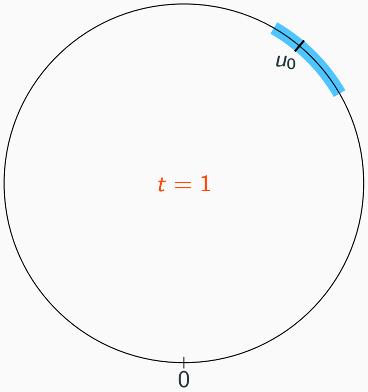

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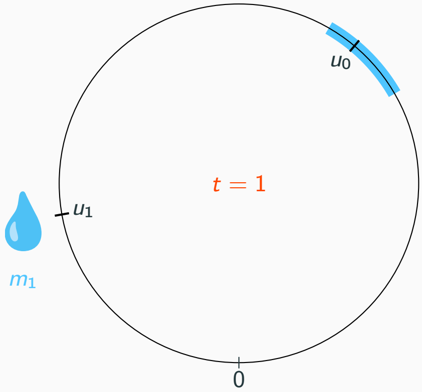

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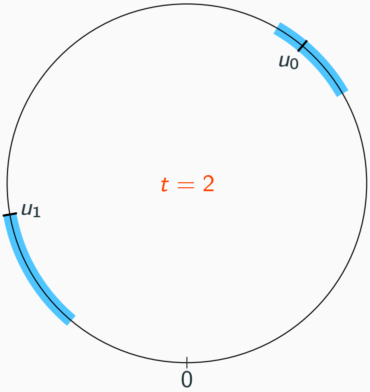

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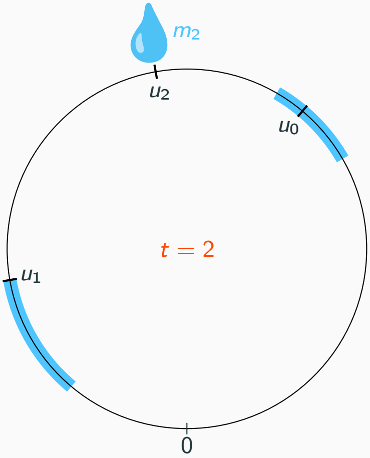

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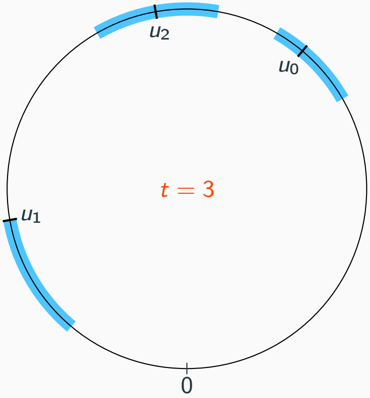

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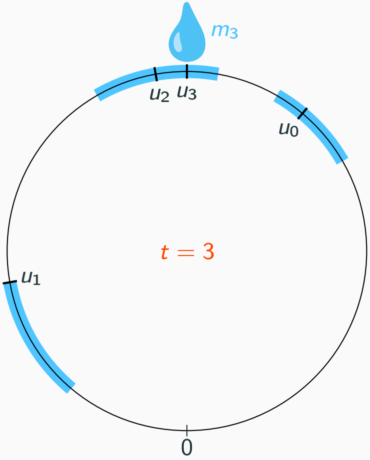

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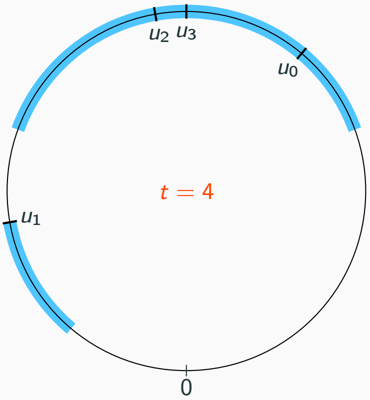

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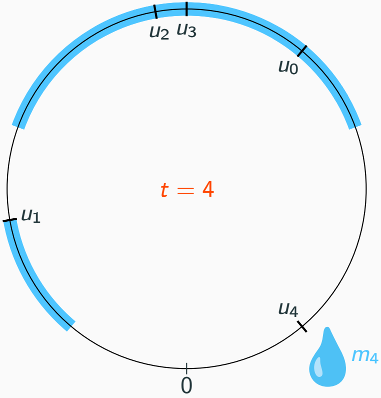

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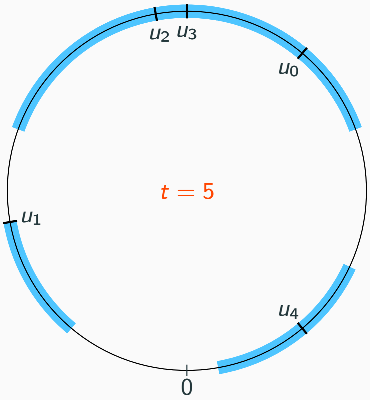

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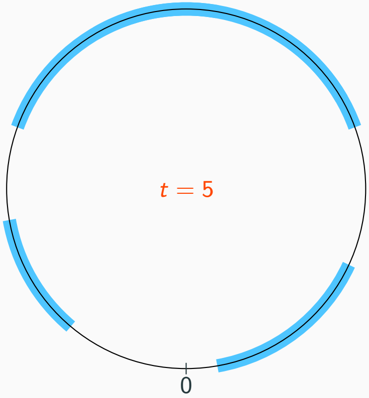

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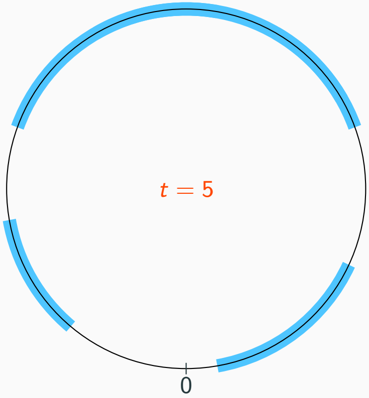

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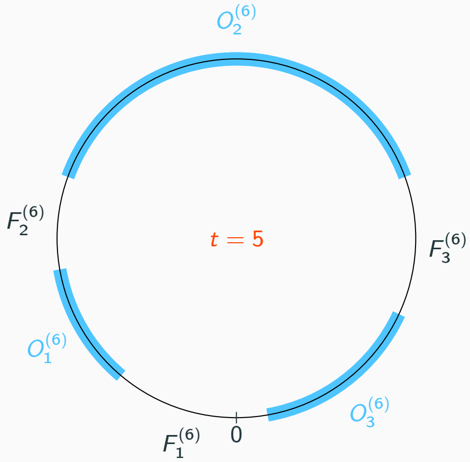

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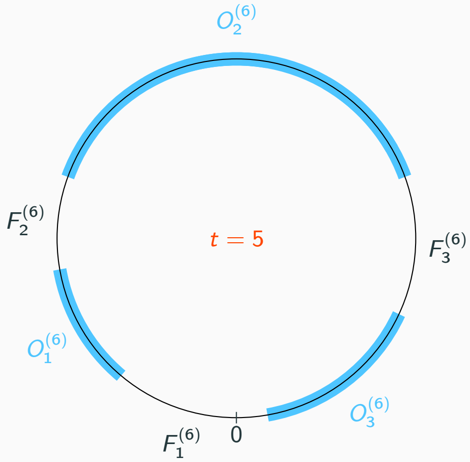
$\Longrightarrow$ $(O_i^{(k)}, F_i^{(k)})_{1 \le i \le N^{(k)}}$ ordered around the circle with $0 \in O_1^{(k)} \cup F_1^{(k)}$



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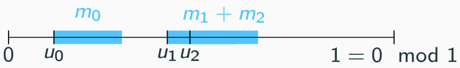
Validity hypotheses:

- dl and dr only depend on what is inside the current component of u_k (one of the $O_i^{(k+\varepsilon)}$)
- invariance by translation of the process

Examples of valid spreading policies

- **Right diffusion at constant speed**: $\overrightarrow{O^{(k)}}, \overrightarrow{F^{(k)}}$ (studied by Bertoin, Miermont [BM06])



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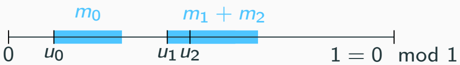
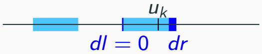


- Diffusion to the closest side of the occupied component (with or without reevaluation)



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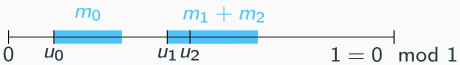

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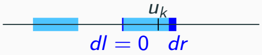

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- Short-sighted jam spreader
- For any ball, pick at random some spreading policy
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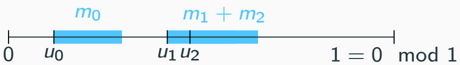

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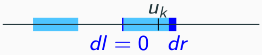

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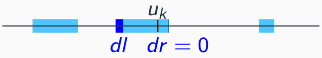
Example that is **not** a valid spreading policy:

- diffusion towards the closest occupied block



A universality result

We fix $m_0, \ldots, m_{k-1}$, with $\sum m_i < 1$. Let $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$. Let $R = 1 - \sum_{i=0}^{k-1} m_i$.

Theorem (Marckert-V. 25+)

Independently of the diffusion policy,

- ***Number of blocks:***
- ***Lengths of the free blocks:***
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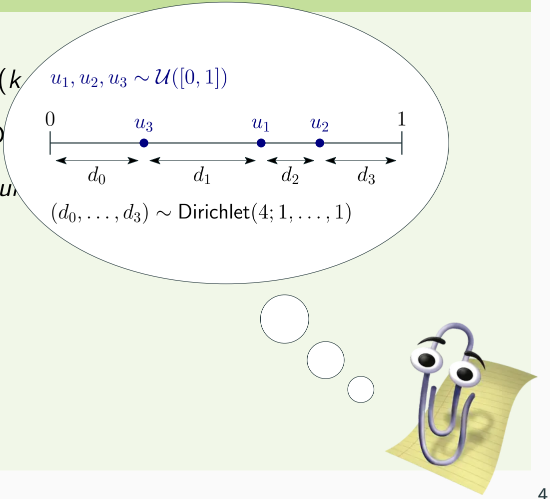

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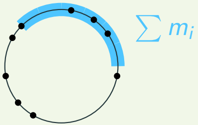

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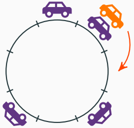
*As a process in k, the following distributions are known and do **not** depend on the dispersion policy:*

- $\mathcal{L}(N^{(k)}, k \geq 0)$
- $\mathcal{L}(\{\{|O^{(k)}|\}\}, k \geq 0)$

Some background on the continuous and discrete parking models

Discrete parking

- introduced by Konheim, Weiss [KW66], studied by Knuth [Knu73]



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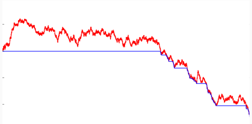

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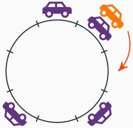

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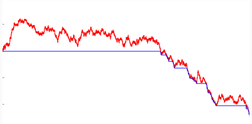

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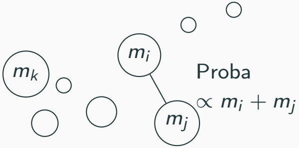

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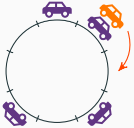

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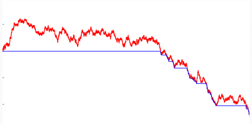

Generalized parking

- Parking on $\mathbb{Z}$ (Przykucki, Roberts, Scott [PRS23])
- Parking on (random) trees (Contat et. al.)
- **Bilateral parking procedures (Nadeau), Golf model on $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$ [Var25]**

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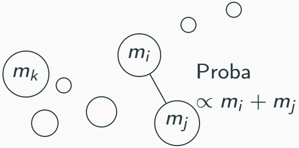

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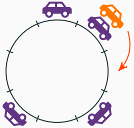

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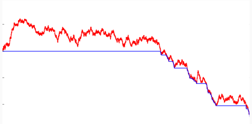

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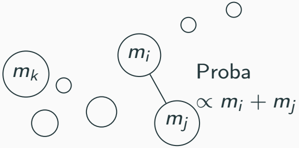

Continuous version of the classical parking

Caravans (Bertoin, Miermont [BM06])

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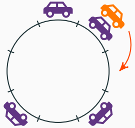

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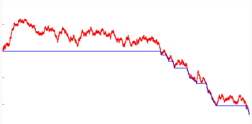


Generalized parking

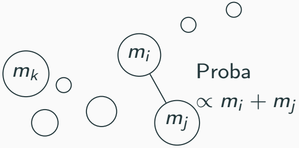
- Parking on $\mathbb{Z}$ (Przykucki, Roberts, Scott [PRS23])
- Parking on (random) trees (Contat et. al.)
- **Bilateral parking procedures (Nadeau), Golf model on $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$ [Var25]**

Continuous version of the classical parking

Caravans (Bertoin, Miermont [BM06])

Additive coalescent

studied by Aldous, Pitman [AP98], Chassaing, Louchard,...



A universality result

We fix $m_0, \dots, m_{k-1}$, with $\sum m_i < 1$. Let $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$. Let $R = 1 - \sum_{i=0}^{k-1} m_i$.

Theorem (Marckert-V. 25+)

Independently of the diffusion policy,

- ***Number of blocks:*** $N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$
- ***Lengths of the free blocks:*** $\frac{\boldsymbol{\sigma}.|F^{(k)}|}{R} \sim \text{Dirichlet}(N^{(k)}; 1, \dots, 1)$
- ***Lengths of the occupied blocks:*** *a formula for* $\mathbb{P}\left(|O^{(k)}| = (M_0, \dots, M_{b-1})\right)$

One great principle

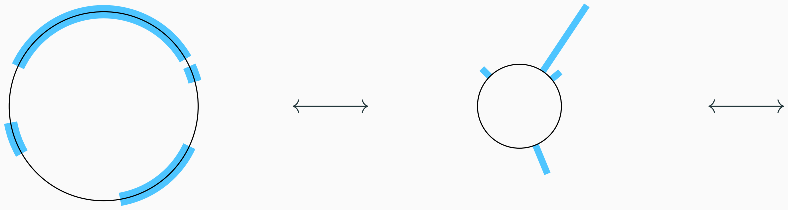
Consider 4 uniform points on $[0, 1]$.

Conditional on :

Then:

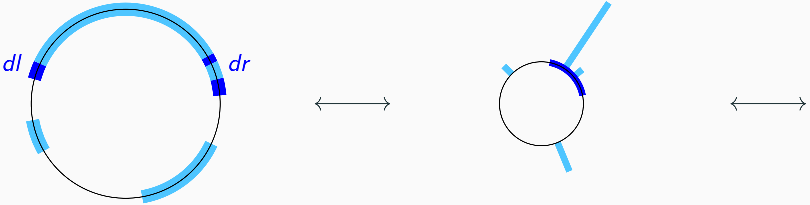
4 uniform points in 3 uniform points in

Peak representation:



Main idea of the proof

Peak representation:



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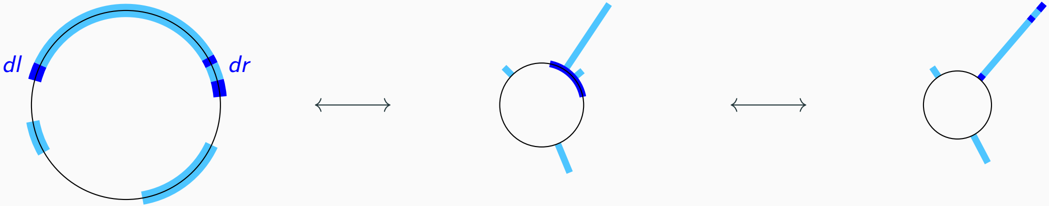

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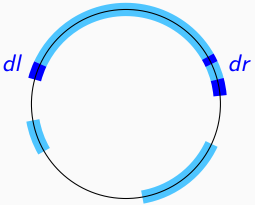
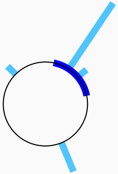


$\longleftrightarrow$



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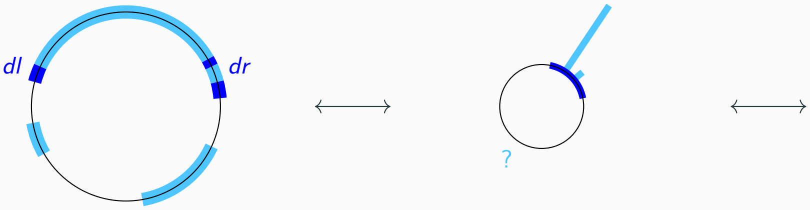

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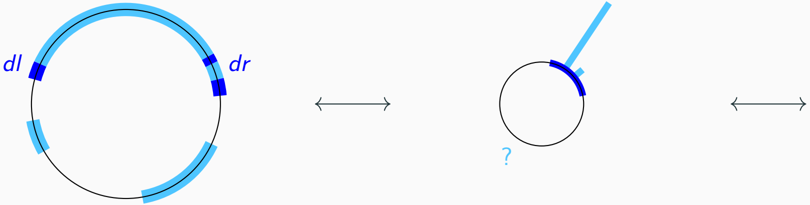

Properties that are **invariant** throughout the dispersion:

- the positions of the peaks are uniform on the smaller cycle $\mathcal{C}_R$ of size $R = 1 - \sum m_i$
- during the dispersion, the probability that the growing peak coalesces with other peaks depends only on the size of the drop

→ the distributions of the peaks' **number**, **heights** and **positions** do not depend on the diffusion policy

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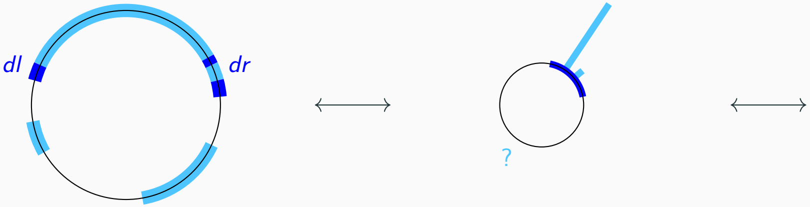

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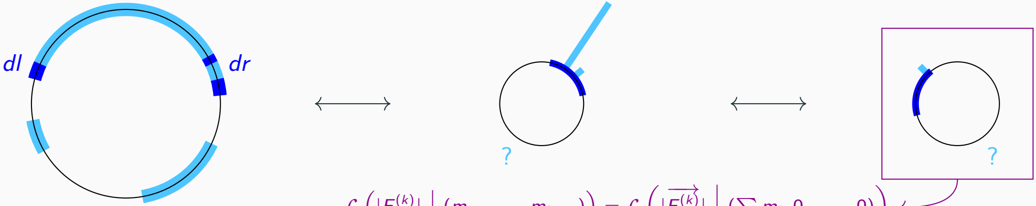

$$\mathcal{L}\left(|F^{(k)}| \;\Big|\; (m_0, \cdots, m_{k-1})\right) = \mathcal{L}\left(|\overrightarrow{F^{(k)}}| \;\Big|\; (\sum m_i, 0, \cdots, 0)\right)$$

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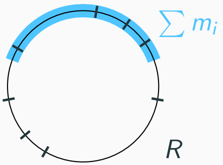
→ the distributions of the peaks' **number**, **heights** and **positions** do not depend on the diffusion policy

→ even more surprisingly, the peaks **number** and **positions** do not depend on which peak is extended by the diffusion

Distribution of the number of blocks $N^{(k)}$

$$\mathcal{L}\left(|F^{(k)}| \;\middle|\; (m_0, \cdots, m_{k-1})\right) = \mathcal{L}\left(|\overrightarrow{F^{(k)}}| \;\middle|\; (\sum m_i, 0, \cdots, 0)\right)$$



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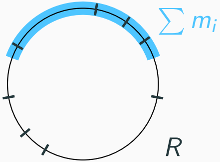

$$N^{(k)} = 1 + \sum_{j=1}^{k-1} \mathbb{1}_{u_k \notin \blacksquare}$$

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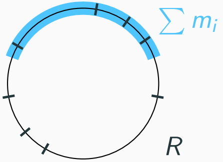
Theorem (Distribution of $N^{(k)}$)

Let $B(k-1, R) \sim \text{Binomial}(k-1, R)$, *then*

$$N^{(k)} \overset{(d)}{=} 1 + B(k-1, R)$$



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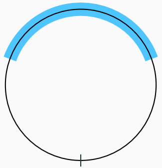
Distribution of the occupied blocks

One block case:

$$\mathbb{P}\left(N^{(k)}=1\right)=\left(\sum m_i\right)^{k-1}=:Q\left(\sum m_i,k\right)$$



and, conditional on $N^{(k)}=1$, $O^{(k)}$ is reduced to an interval $[A,A+\sum m_i]$ with A uniform on $\mathcal{C}$

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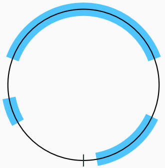

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General case:

Theorem

$$\mathbb{P}\left(|O^{(k)}| = (M_0, \dots, M_{b-1})\right) = \mathrm{T}(M_0, \dots, M_{b-1}) \sum_{P \in \mathcal{P}(k,b)} \left[\prod_{\ell=0}^{b-1} Q(M_j, |P_j|) \, \mathbb{1}_{\sum_{i \in P_\ell} m_i = M_\ell}\right]$$

where

- $\mathcal{P}(k, b)$ *is the set of partitions* $P = (P_0, \dots, P_{b-1})$ *of* $\{1, \dots, k-1\}$ *into* b *non empty parts,*

- $\mathrm{T}(M_0, \dots, M_{b-1}) = M_0 \frac{(1-\Sigma M_\ell)^{b-1}}{(b-1)!} + \frac{(1-\Sigma M_\ell)^b}{b!}$.

Summary of universality results

Theorem

For any continuous model with valid spreading policy, the following distributions are ***explicit*** *and* ***independent of the spreading policy****:*

- *With k fixed:*
  - $\mathcal{L}(O^{(k)}, F^{(k)})$
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Corollary: results on $O^{(k)}, F^{(k)}$ for one spreading policy are valid for any spreading policy !

Asymptotic results

With n (random) masses, $n \to \infty$, for example if

- $\forall i, m_i = 1/n$ and consider the process until time n
- $\forall i, m_i = \ell_i/n$ (where ℓ_i are i.i.d. with $\mathbb{E}[\ell_i] < \infty$ and satisfy some regularity assumption), until time $t = \sup\{k : \sum_{i=0}^{k} m_i < 1\}$.

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Corollary (Bertoin, Miermont [BM06]; Marckert, V. 25+)

There exists a limit process S such that

$$\left(\frac{\text{LargestBlock}^{(\text{i})}}{n}, 1 \leq i \leq j\right) \xrightarrow[n\to\infty]{(d)} (\text{SortedExc}(S)_i, 1 \leq i \leq j)\,.$$

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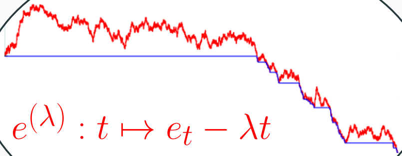

$$e^{(\lambda)} : t \mapsto e_t - \lambda t$$

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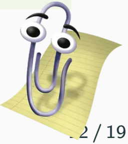
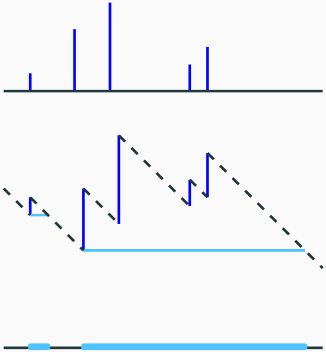


Key tool to encode the initial configuration: the “collecting path”

Illustration:



Definition: $S_x = -x + \sum_{j=0}^{\lfloor n-\lambda\sqrt{n}\rfloor} m_j \mathbb{1}_{u_j \le x}, \ \forall x \in [0,1]$

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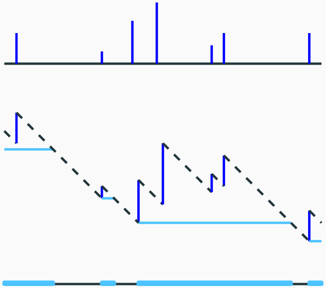

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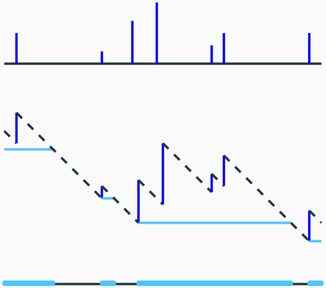

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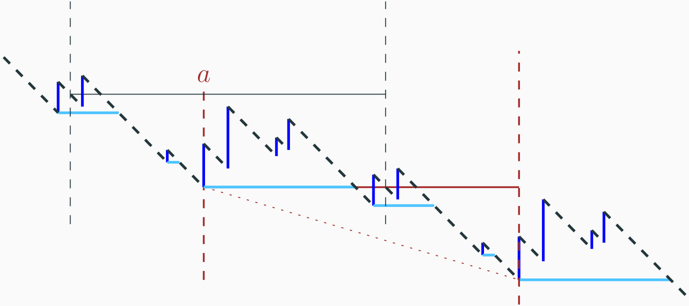
Periodic extension $\bar{S}$:



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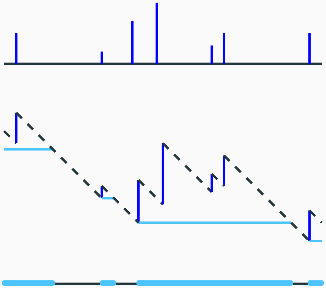

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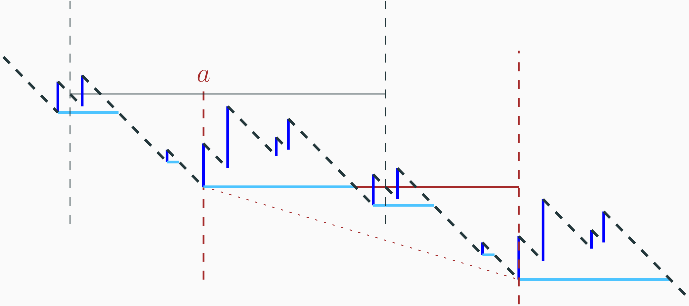


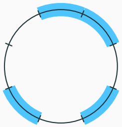
Convergence: $\bar{S}_{[a,a+1]}$ converges (in distribution) to $e^{(\lambda)}$, and

$$\left(\text{SortedExc}(\bar{S}_{[a,a+1]})_i\right)_{1\leq i\leq j} \xrightarrow[n\to\infty]{(d)} \left(\text{SortedExc}(e^{(\lambda)})_i\right)_{1\leq i\leq j}.$$

Discrete space and discrete masses

New definition on $\mathcal{C}_n := \{0/n, \cdots, (n-1)/n\} \subset \mathcal{C}$:

- masses arrive on $\mathcal{C}_n$: $\forall i, u_i \sim \mathcal{U}(\mathcal{C}_n)$
- they cover intervals with extremities in $\mathcal{C}_n$
- their spreading policy is only invariant by $1/n$ rotation



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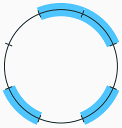

Theorem (Similar universality result)

For any discrete model with valid spreading policy, the following distributions are ***explicit*** *and* ***independent of the spreading policy****:*

- *With k fixed:*
  - $\mathcal{L}(O^{(k)}, F^{(k)})$
  - $\mathcal{L}\left(|F^{(k)}| \;\middle|\; (m_0, \cdots, m_{k-1})\right) = \mathcal{L}\left(|\overrightarrow{F^{(k)}}| \;\middle|\; (\sum m_i, 0, \cdots, 0)\right)$
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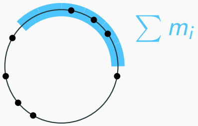
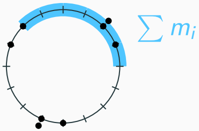
Discrete VS Continuous model

Continuous process:



$$N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$$

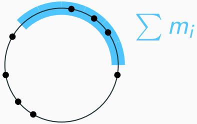
Discrete process:



$$N^{(k,n)} \overset{(d)}{=} 1 + \ldots$$

Discrete VS Continuous model

Continuous process:



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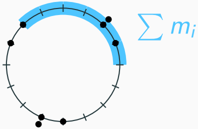

$N^{(k,n)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R-1/n)$

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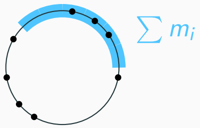

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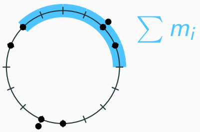

$$N^{(k,n)} \overset{(d)}{=} 1 + \text{NbDiff}(\text{Binomial}(k-1, R-1/n))$$

Discrete VS Continuous model

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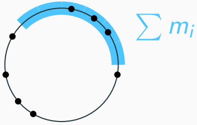

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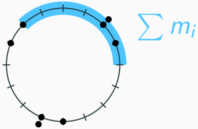

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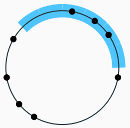
Asymtotic behavior when $k = n - \lambda\sqrt{n}$ and $\forall i, m_i = 1/n$

Number of blocks: $\frac{N^{(k)}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda$

Number of blocks: $\frac{N^{(k,n)}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda(1-e^{-1})$

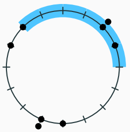
Discrete VS Continuous model

Continuous process:



$N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$

Discrete process:



$N^{(k,n)} \overset{(d)}{=} 1 + \text{NbDiff}(\text{Binomial}(k-1, R-1/n))$

Asymtotic behavior when $k = n - \lambda\sqrt{n}$ and $\forall i, m_i = 1/n$

Number of blocks: $\frac{N^{(k)}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda$

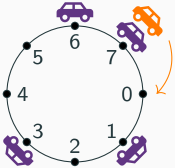
Number of blocks: $\frac{N^{(k,n)}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda(1-e^{-1})$

Large block sizes [BM06, CL02]: $\left(\frac{\text{LargestBlock}^{(\mathbf{i})}}{n}, 1 \le i \le j\right) \xrightarrow[n\to\infty]{(d)} (\text{SortedExc}(e^{(\lambda)})_i, 1 \le i \le j)$

Construction cost of discrete models

The parking model

The parking model:



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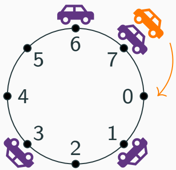

Free vertices: $N_F = n - \#$

LargestBlock$^{(i)}$ = size of the ith largest block of occupied vertices.

The parking model

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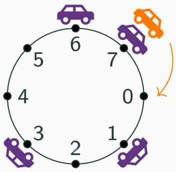

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Theorem [Pittel 87, Chassaing-Louchard 02]

- If $N_F = N_F(n) \sim an$, $a > 0$, then LargestBlock$^{(1)}$ converges in probability:
$$\text{LargestBlock}^{(1)} = \frac{\log n - 3/2 \log\log n}{a - 1 - \log a} + O(1).$$
- If $N_F \ll \sqrt{n}$, then $\frac{\text{LargestBlock}^{(1)}}{n} \xrightarrow{\mathbb{P}} 1$.
- If $N_F \gg \sqrt{n}$, then $\frac{\text{LargestBlock}^{(1)}}{n} \xrightarrow{\mathbb{P}} 0$.
- Description of the **phase transition**: si $N_F/\sqrt{n} \to \lambda \geq 0$,
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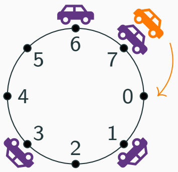

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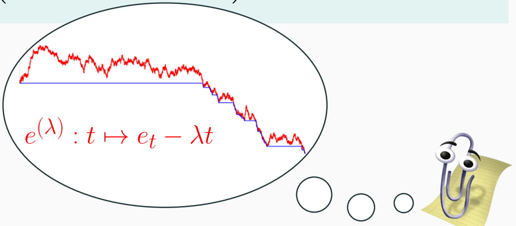

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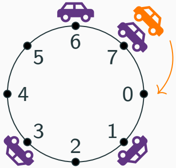

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Generalised parking model: local and invariant under rotation parking policy (Nadeau 23)

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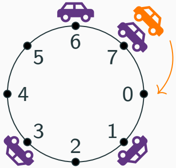

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Theorem (V. 25 - Universality of the distribution of the set of free vertices H^F for the generalised parking)

$$\mathbb{P}\left(\mathrm{H}^F = X\right) = \frac{1}{n^{n-N_F}} \binom{n - N_F}{\ell_1, \ldots, \ell_{N_F}} \prod_{i=1}^{N_F} (\ell_i + 1)^{\ell_i - 1}$$

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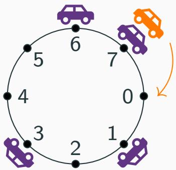

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Construction cost of the parking

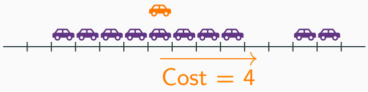
Definition of the cost

$$\mathrm{GlobalCost}_n(r) = \sum_{i=1}^{r} \mathrm{Cost}^i_{B_i}$$

where

- B_i is the size of the block in which the ith car falls
- $\mathrm{Cost}^i_{B_i}$ is a random variable whose distribution depends only on B_i

For the parking:



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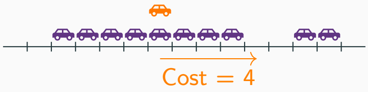

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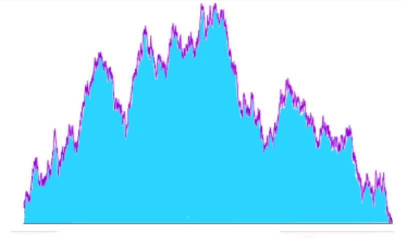
If $r = n$, (Flajolet-Poblete-Viola 98)

$$\frac{\mathrm{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e_t \, dt$$

Illustration de $\int_0^1 e_t \, dt$:



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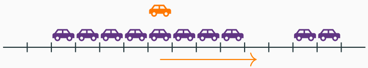

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$$\frac{\text{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e^{(\lambda)}(t) - \inf_{s\le t} e^{(\lambda)}(s) dt =: F(\lambda)$$

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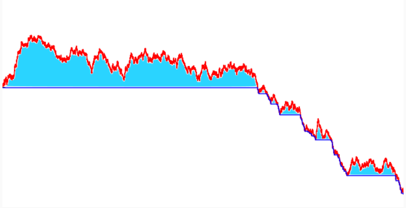

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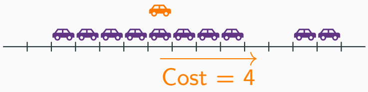

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Theorem (Generalised parking - Marckert-V.25+)

Convergence in distribution of

$$\frac{GlobalCost_n\left(\lfloor n - \lambda\sqrt{n} \rfloor\right)}{\sqrt{n}\alpha_n}$$

towards an explicit limit; under some hypotheses on $\mathbb{E}[Cost_k]$ *and* $Var(Cost_k)$

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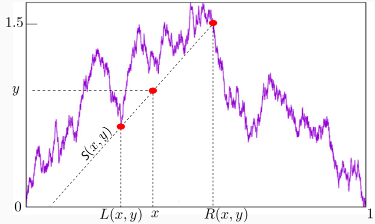

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When cars do random walks of parameter p

If $p \neq 1/2$,

$$\frac{\mathrm{GlobalCost}_n(\lfloor n - \lambda\sqrt{n} \rfloor)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \frac{1}{|2p-1|} F(\lambda)$$

If $p = 1/2$,

$$\frac{\mathrm{GlobalCost}_n(\lfloor n - \lambda\sqrt{n} \rfloor)}{n^{5/2}} \xrightarrow[n\to\infty]{(d)} \frac{1}{3} G(\lambda)$$

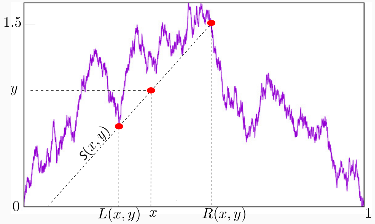
where $G(\lambda) = \int_0^1 \int_0^{e_x} (R(x,y) - L(x,y)) \mathbb{1}_{S(x,y)\geq\lambda}\, dy\, dx$

Partial idea of the proof

- Measure encoding the size of the blocks :
$M^{(n,\lambda)} = \frac{1}{\sqrt{n}} \sum_{k \leq n - \lambda\sqrt{n}} \delta_{B_k/n}$

$$M^{(n,\lambda)} \overset{(d)}{\to} M_\lambda$$

(for the vague topology on the set of Borelian measures on $(0,1)$)



- If f is such that $\mathbb{E}[\mathrm{Cost}_k] \underset{k\to\infty}{\sim} f(k)$, then

$$\frac{\mathrm{GlobalCost}_n\left(\lfloor n - \lambda\sqrt{n}\rfloor\right)}{\sqrt{n}\alpha_n} \sim \frac{1}{\sqrt{n}\alpha_n} \sum_{k \leq n - \lambda\sqrt{n}} f(B_k) = \langle f, M^{(n,\lambda)}\rangle \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \langle f, M_\lambda\rangle$$

where $\langle f, M_\lambda(e)\rangle = \int_0^1 \int_0^{e_x} \frac{2}{R(x,y) - L(x,y)} f\Big(R(x,y) - L(x,y)\Big) \mathbb{1}_{S(x,y)\geq\lambda} dydx.$

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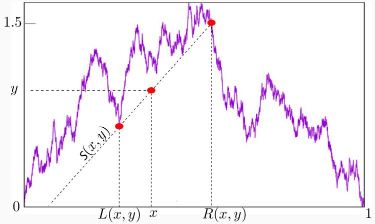

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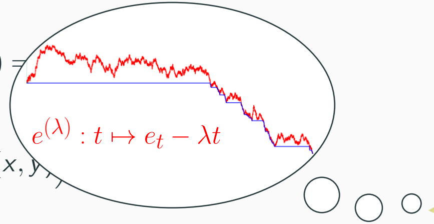

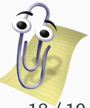

💧 **Thank you !** 💧

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