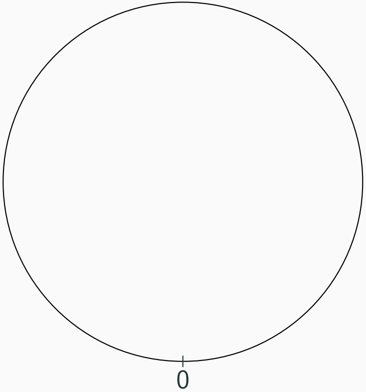
Stochastic diffusion model on the unit circle

Zoé Varin

28 August 2024

Joint work with Jean-François Marckert

Definition of the model



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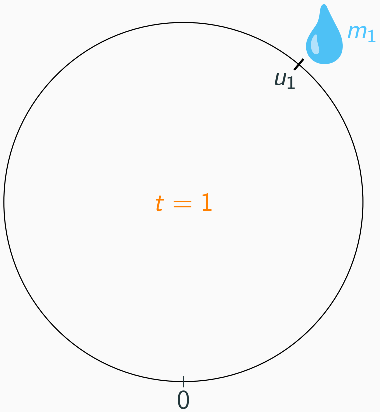

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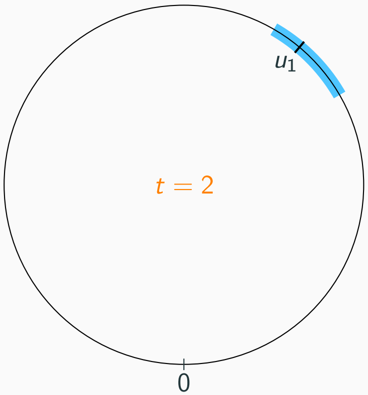

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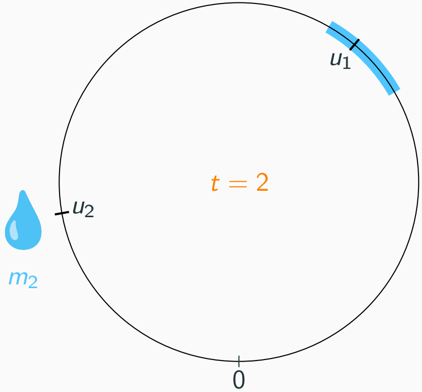

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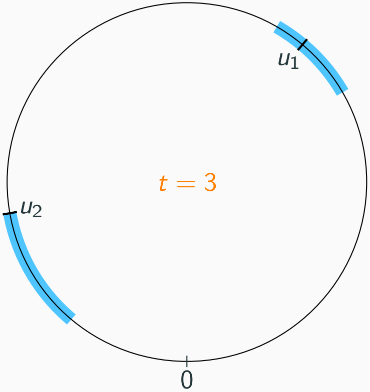

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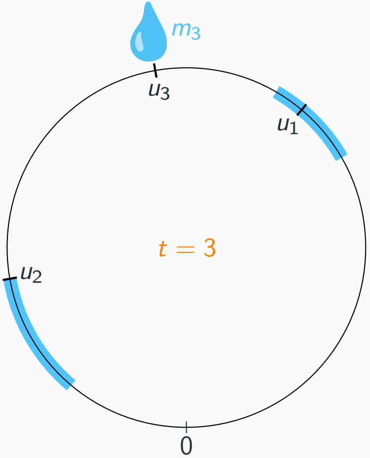

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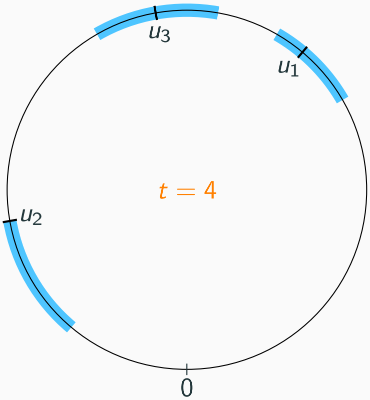

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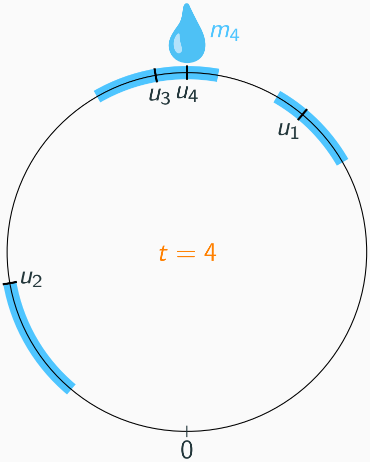

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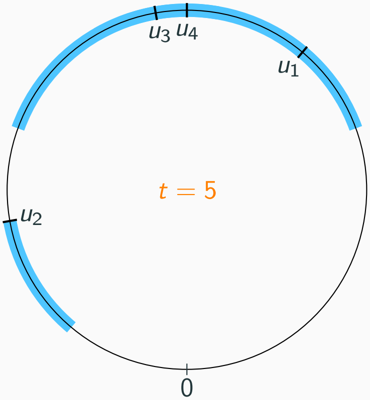

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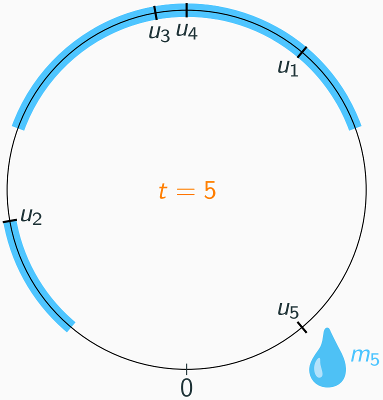

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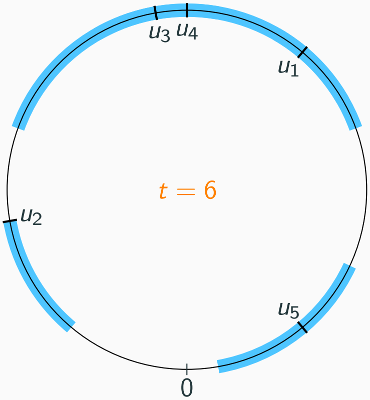

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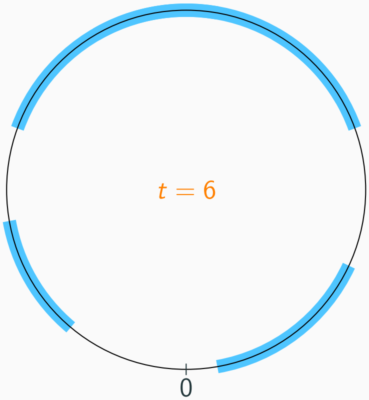

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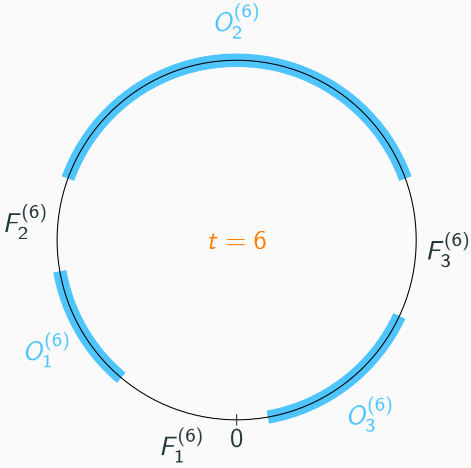

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- more precisely, $N^{(k)}$ **blocks** of each type

$\implies$ $(O_i^{(k)}, F_i^{(k)})_{1 \leq i \leq N^{(k)}}$ ordered around the circle with $0 \in O_1^{(k)} \cup F_1^{(k)}$

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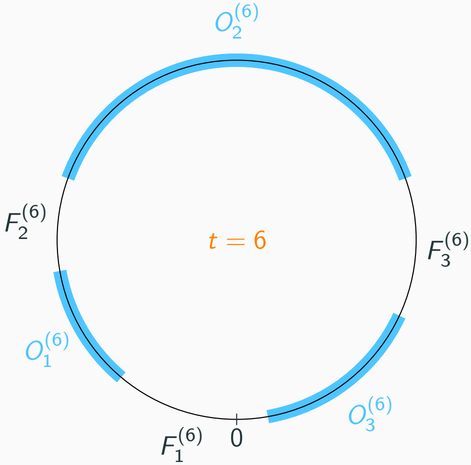

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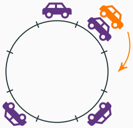

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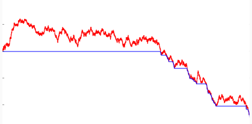

Some background on the continuous and discrete parking models

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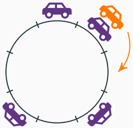

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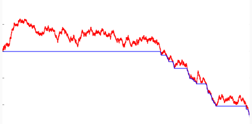
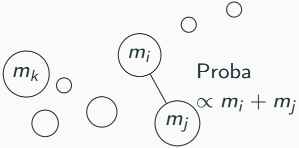


Additive coalescent

studied by Aldous, Pitman [AP98], Chassaing, Louchard,...



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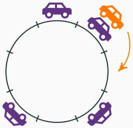

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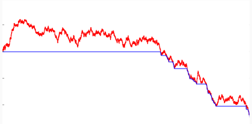

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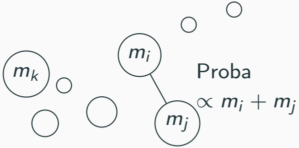

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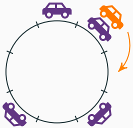

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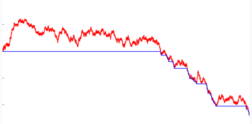

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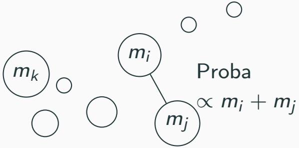

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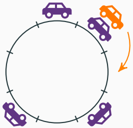

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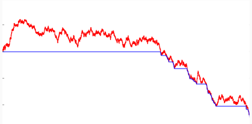

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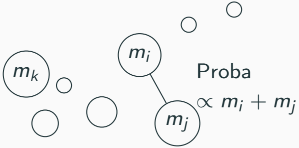
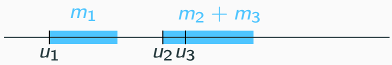


Examples of spreading policies

- **Right diffusion at constant speed**: $\overrightarrow{O^{(k)}}, \overrightarrow{F^{(k)}}$



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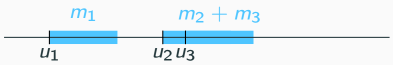

- Proportion $(p, 1-p)$ to the right/left, with $p \sim \mathcal{U}([0,1])$
- Diffusion to the closest side (with or without constant reevaluation)
- Infinitesimal particle like diffusion
- for any ball, pick at random some spreading policy
- ...

What is a reasonable spreading ?

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Reasonable **hypotheses:**

- dl and dr only depend on what is inside the current component of u_k (one of the $O_i^{(k+\varepsilon)}$)
- invariance by translation of the process

A universality result

We consider $\frac{\sigma.|F|}{R} = \left(\frac{|F_{\sigma_i}|}{R}\right)_{1\leq i\leq N^{(k)}}$, for any $\sigma \in \mathfrak{S}_{N^{(k)}}$ and let $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$.

Theorem

Independently of the diffusion policy,

- ***Number of blocks:*** $N^{(k+1)} \overset{(d)}{=} 1 + Binomial(k-1, 1-\sum_{i=1}^{k} m_i)$

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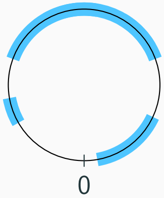
Corollary

Asymptotic results *(for independent random masses): convergence of the block lengths process (time-changed) to an eternal additive coalescent (Bertoin, Miermont [BM06])*

Size bias and exchangeability of the blocks I

Invariance by rotation of $B^{(k)} = (O^{(k)}, F^{(k)})$ as subsets of $\mathcal{C}$

Size bias of the sequence $Seq(B^{(k)}) = (O_i^{(k)}, F_i^{(k)})_{1 \leq i \leq N^{(k)}}$ since $0 \in O_1^{(k)} \cup F_1^{(k)}$



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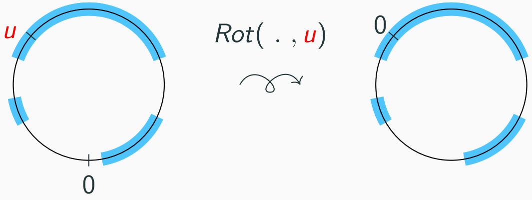
$$\mathcal{L}\left(B^{(k)}\right) = \mathcal{L}\left(Rot(B^{(k)}, u)\right) \qquad \text{with } u \sim \mathcal{U}(\mathcal{C})$$



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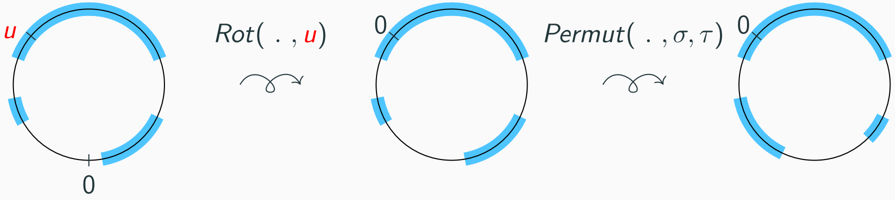


Size bias and exchangeability of the blocks II

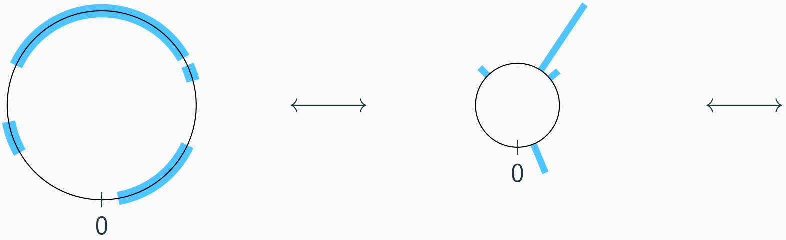
Reminder: $B^{(k)} = (O^{(k)}, F^{(k)})$

Theorem (Exchangeability)

- *The distribution of* $B^{(k)}$ *is independent from the diffusion policy.*
- *Concerning the free blocks,*

$$\mathcal{L}\left(|F^{(k)}| \;\middle|\; (m_1, \cdots, m_{k-1})\right) = \mathcal{L}\left(|\overrightarrow{F^{(k)}}| \;\middle|\; \left(\sum m_i, 0, \cdots, 0\right)\right)$$

Peaks representation:



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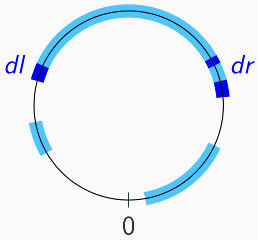

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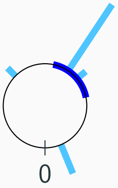

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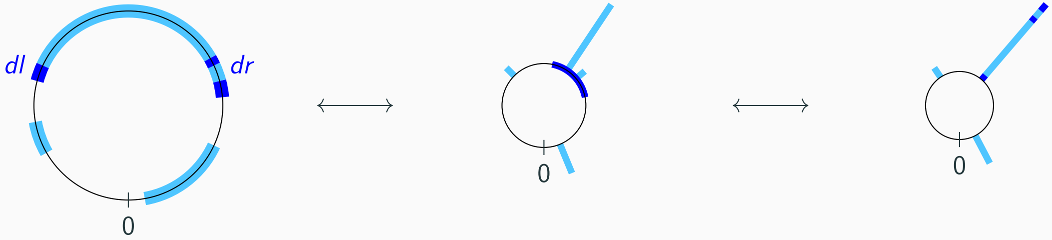

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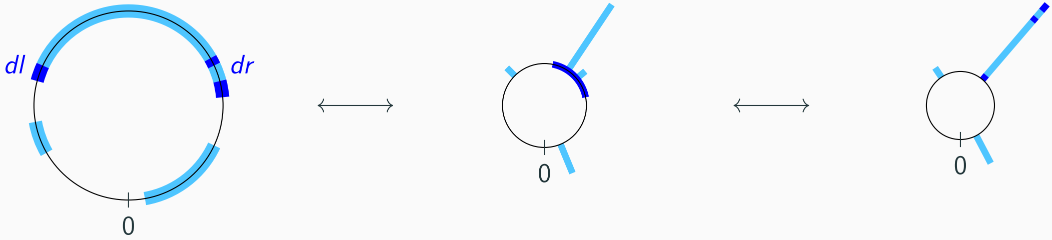

allow to show that, at every step:

- the positions of the peaks are uniform on the smaller cycle $\mathcal{C}_R$ of size $R = 1 - \sum m_i$
- the distributions of the peaks **number**, **lengths** and **positions** do not depend on the diffusion policy

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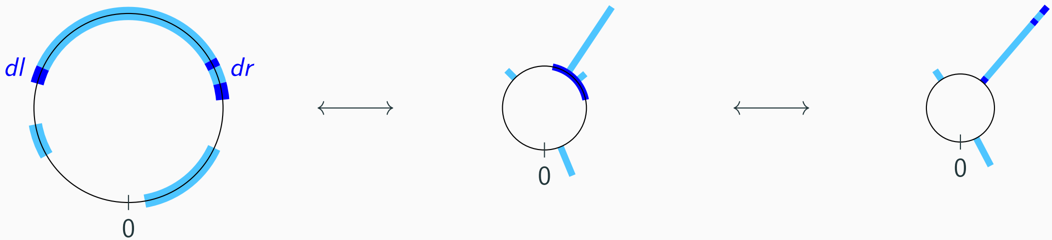
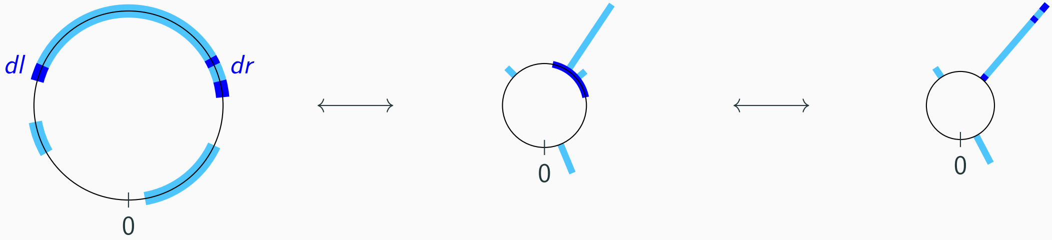


allow to show that, at every step:

- the positions of the peaks are uniform on the smaller cycle $\mathcal{C}_R$ of size $R = 1 - \sum m_i$
- the distributions of the peaks **number**, **lengths** and **positions** do not depend on the diffusion policy
- even more surprisingly, for the peaks **number** and **positions**: do not depend on which peak is extended by the diffusion

Intuition on exchangeability

Peaks representation:



allow to show that, at every step:

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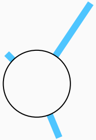
Distribution of the number of blocks $N^{(k)}$ and of the lengths of the free blocks



Reminder: $|F| = (|F_i|)_{1 \leq i \leq N^{(k)}}$ and $\frac{\sigma.|F|}{R} = \left(\frac{|F_{\sigma_i}|}{R}\right)_{1 \leq i \leq N^{(k)}}$, for any $\sigma \in \mathfrak{S}_{N^{(k)}}$.

Theorem (Distribution of $|F|$ conditional on $N^{(k)}$)

Recall that $R = 1 - \sum m_i$.

- ***unbiased version:*** *if* $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$, $\frac{\boldsymbol{\sigma}.|F|}{R} \sim \mathrm{Dirichlet}(N^{(k)}; 1, \cdots, 1)$

Distribution of the number of blocks $N^{(k)}$ and of the lengths of the free blocks

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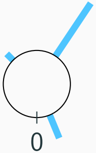

Theorem (Distribution of $|F|$ conditional on $N^{(k)}$)

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- ***unbiased version:*** *if $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$, $\frac{\boldsymbol{\sigma}.|F|}{R} \sim \mathrm{Dirichlet}(N^{(k)}; 1, \cdots, 1)$*
- ***biased version:***

$$\frac{|F|}{R} \sim \begin{cases} \mathrm{Dirichlet}(N^{(k)}; 1, \cdots, 1) & \textit{with probability } 1 - R \\ \mathrm{Dirichlet}(N^{(k)}; 2, 1, \cdots, 1) & \textit{with probability } R \end{cases}$$

Distribution of the number of blocks $N^{(k)}$ and of the lengths of the free blocks

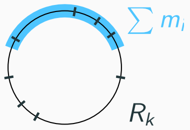
Reminder:

$$\mathcal{L}\left(|F^{(k)}| \;\middle|\; (m_1, \cdots, m_{k-1})\right) = \mathcal{L}\left(|\overrightarrow{F^{(k)}}| \;\middle|\; (\sum m_i, 0, \cdots, 0)\right)$$

Theorem (Distribution of $N^{(k)}$)

If we let $R_k = 1 - \sum_{i=1}^{k} m_i$ and $B(k-1, R_k) \sim Binomial(k-1, R_k)$,

$$N^{(k+1)} \overset{(d)}{=} 1 + B(k-1, R_k)$$

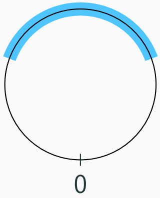


Distribution of the occupied blocks

One block case:

$$\mathbb{P}\left(N^{(k)} = 1\right) = \left(\sum m_i\right)^{k-1} =: Q(\sum m_i, k)$$

and, conditional on $N^{(k)} = 1$, $O^{(k)}$ is reduced to an interval $[A, B]$ with A uniform on $\mathcal{C}$



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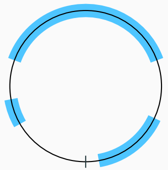

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General case:

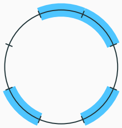
Theorem

$$\mathbb{P}\left(\boldsymbol{\sigma}.|O^{(k)}|=(M_0,\cdots,M_{b-1})\right)=\sum_{P\in P(k,b)}\left[\prod_{\ell=0}^{n-1}\mathbb{1}_{\sum_{i\in P_\ell}m_i=M_\ell}\right]\left[\prod_{\ell=0}^{b-1}Q(M_j,|P_j|)\right]\frac{(1-\sum m_i)^{b-1}}{(b-1)!}$$

where $\boldsymbol{\sigma}\sim\mathcal{U}(\mathfrak{S}_{N^{(k)}})$ *and* $P(k,b)$ *is the set of partitions* $P=(P_0,\cdots,P_{b-1})$ *of the set* $\{1,\cdots,k-1\}$ *as a sequence of* b *non empty parts.*

Conclusion and perspectives

- discrete space extension



- analysis of the asymptotic cost of parking procedures

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