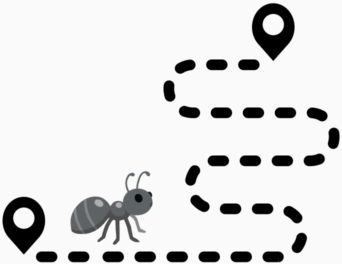
A probabilistic reinforcement-learning algorithm to find shortest paths in a graph

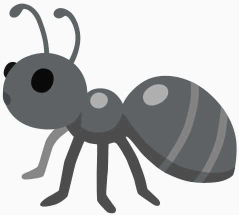
Zoé Varin

December 8th, 2025

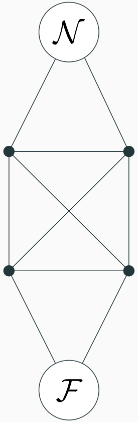
Joint work with Cécile Mailler



Introduction



Definition of the model (one-nest version)



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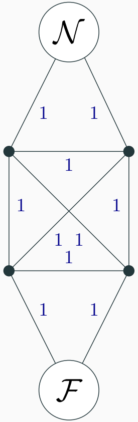

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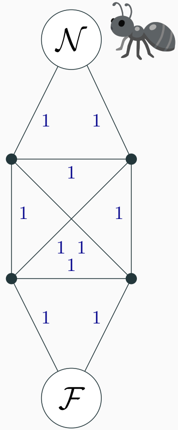

At each step n:

- **random walk X** weighted by $W(n)$:

$$\mathbb{P}\left(u \to v\right) = \frac{W_{uv}(n)}{\sum_{e:u\in e} W_e(n)}$$

starting from $\mathcal{N}$, stopped at $\mathcal{F}$

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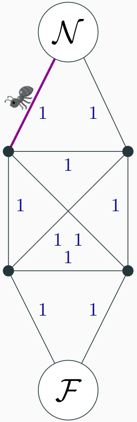

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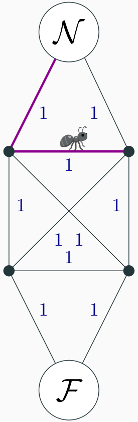

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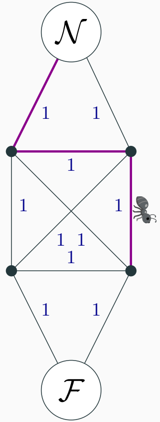

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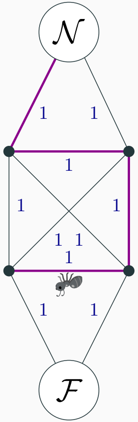

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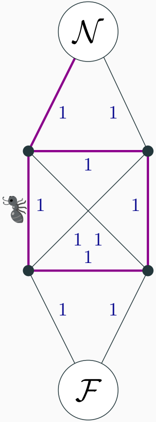

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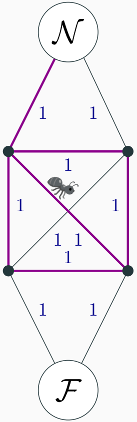

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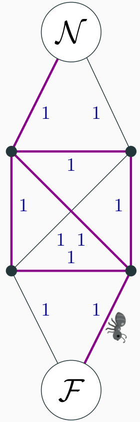

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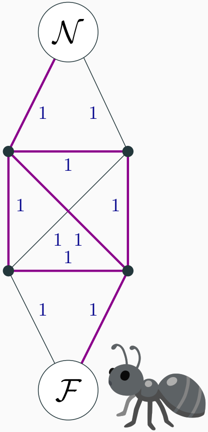

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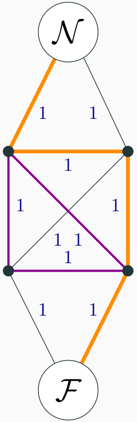

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$$\forall e, W_e(n+1) = W_e(n) + \mathbb{1}_{e \in \gamma}$$

Loop-erased (LE) model: $\gamma = LE(X)$

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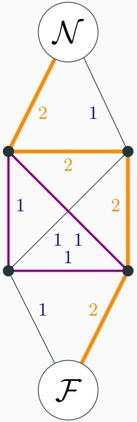

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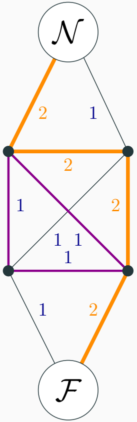

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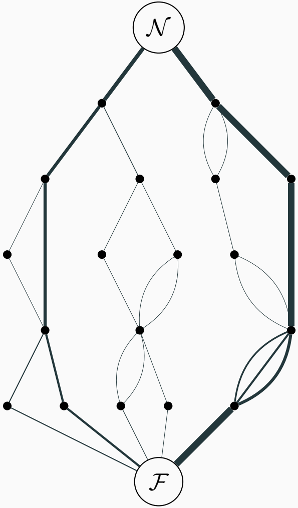

Simulations for $n = 10^8$

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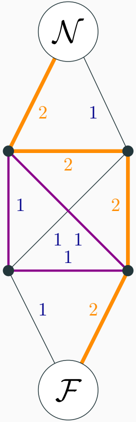

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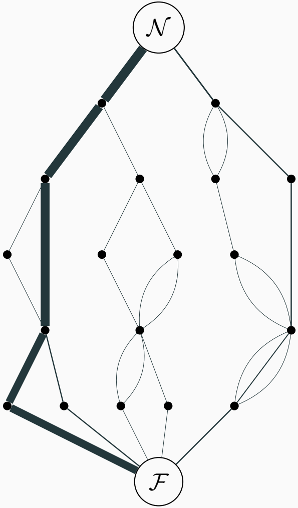

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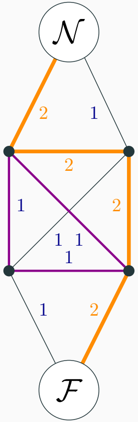

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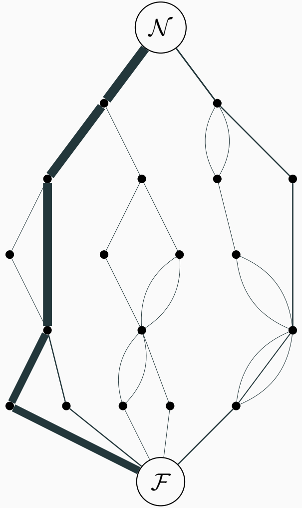
Question: Do the ants find shotest paths from $\mathcal{N}$ to $\mathcal{F}$?
$\rightarrow$ Does $\left(\frac{W_e(n)}{n}\right)_e$ converge ? Towards which limit ?



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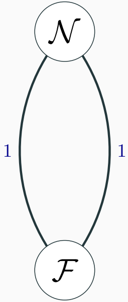
A warm-up and a Pólya urn



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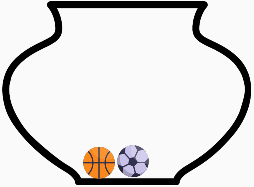

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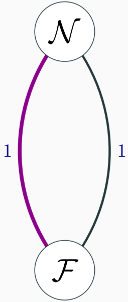

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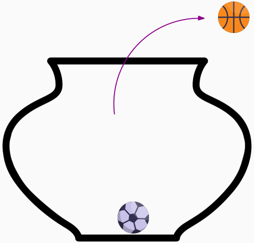

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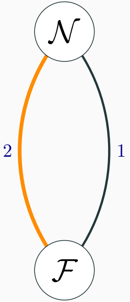

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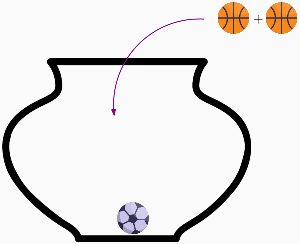

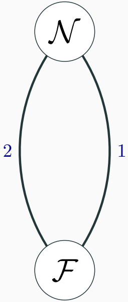

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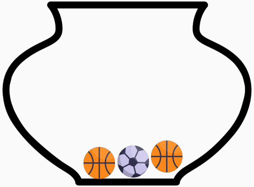

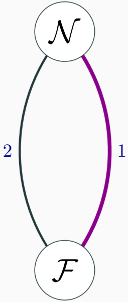

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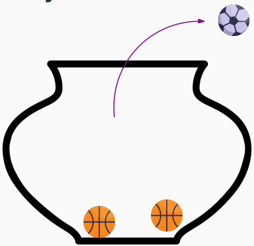

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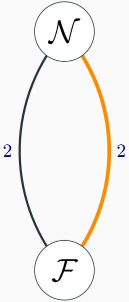

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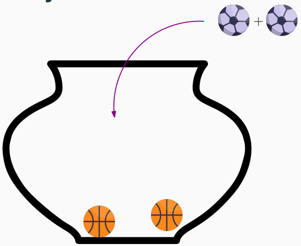

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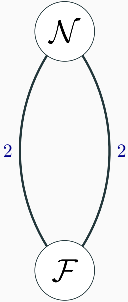

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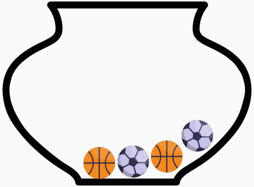

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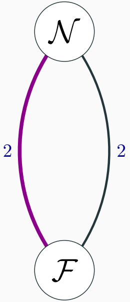

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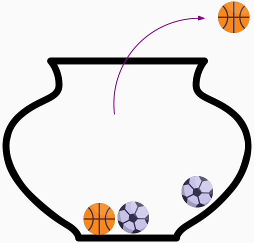

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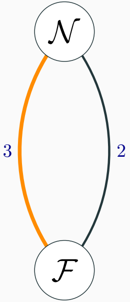

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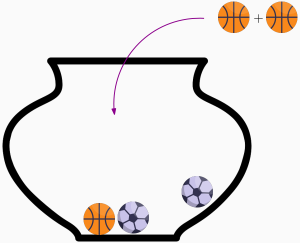

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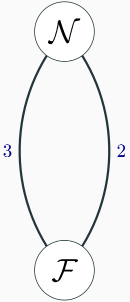

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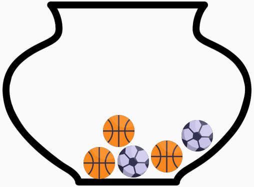

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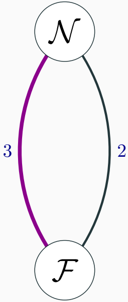

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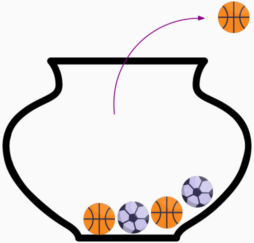

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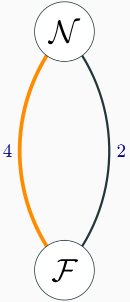

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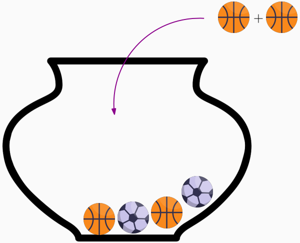

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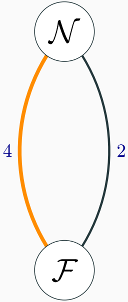

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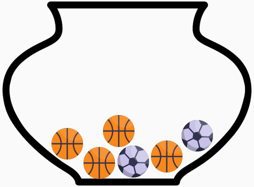

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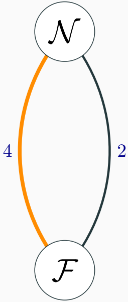

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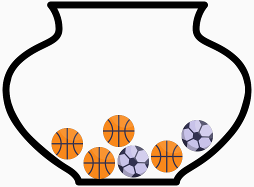
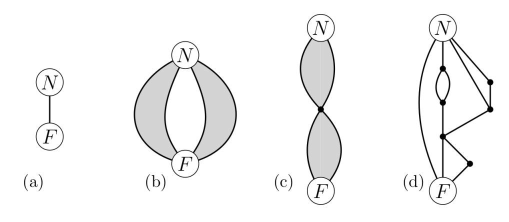


Asymptotic behavior:
Almost surely,

$$\frac{\#\text{🏀}}{n} \xrightarrow[n\to\infty]{} U \sim \mathcal{U}([0,1])$$

Loop-erased (LE) model on series-parallel graphs

Recursive definition of series-parallel (SP) graphs (picture from [KMS22a]):



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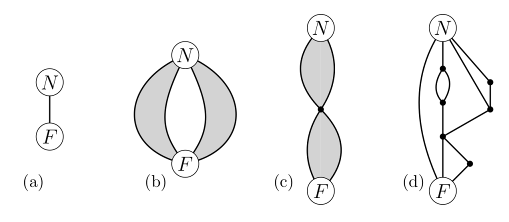

Theorem (Kious, Mailler, Schapira [KMS22a])

If G is a SP graph, then in the loop-erased (LE) model, almost surely,

$$\frac{W_e(n)}{n} \xrightarrow[n\to\infty]{} \chi_e, \quad \forall e \in E$$

where $(\chi_e)_{e\in E}$ is a random vector such that $\forall e$, $\chi_e \neq 0$ if and only if e belongs to a shortest path from N to F.

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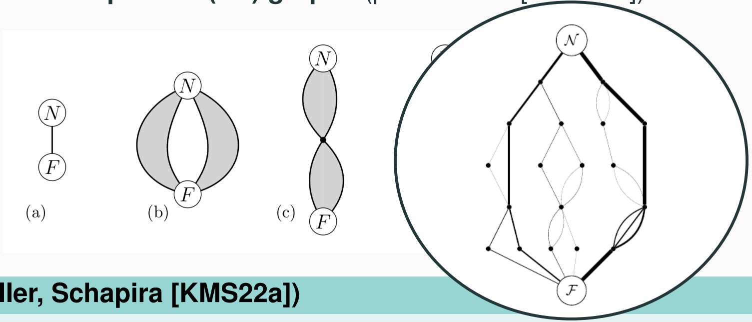

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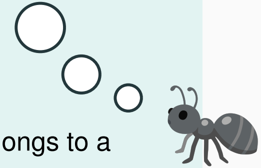
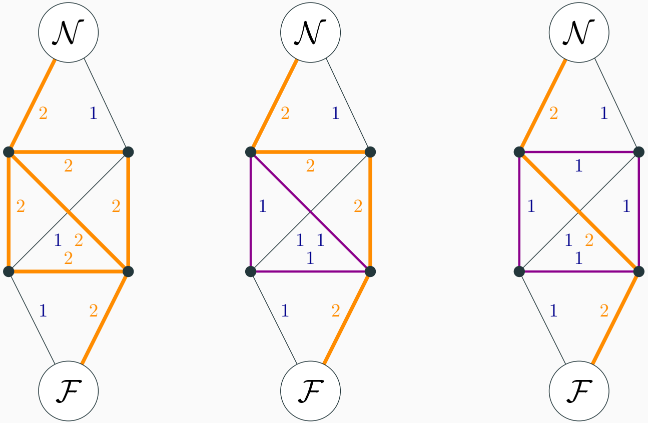


Other reinforcement models



(T) trace: $\gamma = X$

(LE) loop-erased: $\gamma = LE(X)$

(G) geodesic: $\gamma = \text{ShortestPath}(X)$

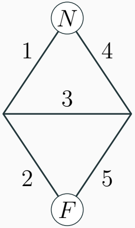
At each step n:

•**random walk** X:

•**depositing pheromones** on γ:

$$\forall e, W_e(n+1) = W_e(n) + \mathbb{1}_{e \in \gamma}$$

The lozenge graph:



Theorem (Kious, Mailler, Schapira [KMS22a])

Almost surely,

$$\frac{W_i(n)}{n} \xrightarrow[n\to\infty]{} \chi_i, \quad \forall 1 \leq i \leq 5$$

where $(\chi_i)_{1\leq i\leq 5}$ is a random vector, such that almost surely,
$\chi_1 = \chi_2 = 1 - \chi_4 = 1 - \chi_5 \in (0,1)$ and $\chi_3 = 0$.

Conjecture for the loop-erased (LE) and geodesic (G) models

Conjecture [KMS22a]

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(LE) model $\chi_e \neq 0$ a.s. **if and only if** e belongs to a shortest path from N to F

(G) model $\chi_e \neq 0$ a.s. **only if** e belongs to a shortest path from N to F

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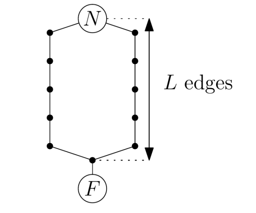
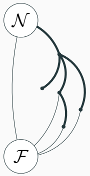


For L large enough, there exists e such that

$$\mathbb{P}\left(W_e(n)/n \to 0\right) > 0$$

Trace (T) model ([KMS22b])

G is *tree-like* if $G \backslash \{\mathcal{F}\}$ is a tree.



Theorem [KMS22b]

If $G = (V, E)$ is *tree-like* and $a = \{\mathcal{N}, \mathcal{F}\} \in E$ with multiplicity 1, then

$$\frac{W_a(n)}{n} \to 1 \quad \text{and} \quad \frac{W_e(n)}{n} \to 0, \quad \forall e \in E \backslash \{a\}$$

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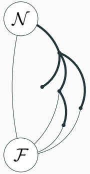

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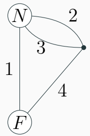
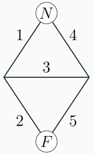
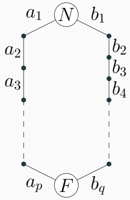
Other examples: the cone, the lozenge and the (p, q)-path





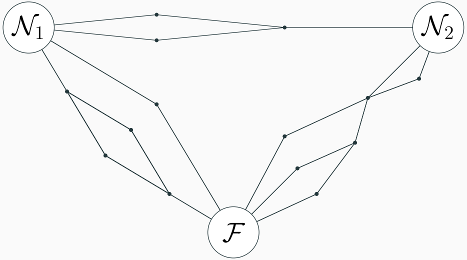


$\frac{W(n)}{n} \xrightarrow[n\to\infty]{} (1, 1/3, 1/3, 0)$ $\frac{W(n)}{n} \xrightarrow[n\to\infty]{} (w^\star, 1/2, 1/2, w^\star, 1/2)$ $\frac{W_{a_k}(n)}{n} \xrightarrow[n\to\infty]{} \alpha^k$, $\frac{W_{b_k}(n)}{n} \xrightarrow[n\to\infty]{} \beta^k$

Conjecture: deterministic limit for any graph without multiple-edges adjacent to F.

Back to the (LE) model: multinest version

2-nest version: at every step n, $\mathcal{N}(n) = \begin{cases} \mathcal{N}_1 & \text{with proba } \alpha \in (0,1) \\ \mathcal{N}_2 & \text{with proba } 1-\alpha \end{cases}$.



Back to the (LE) model: multinest version on triangle-SP graphs

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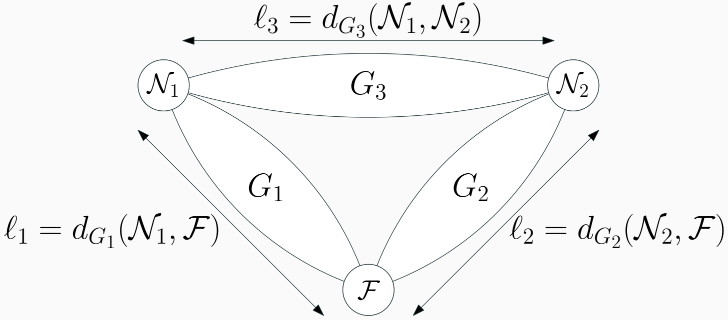
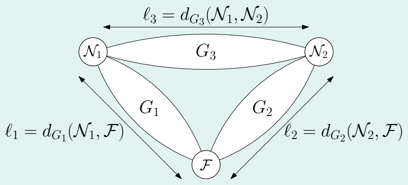


Triangle-SP graph: G_1, G_2, G_3 series-parallel graphs

Our main result: convergence of the 2-nest LE model on triangle-SP graphs

$N_i(n)$ = number of steps at which edges in G_i have been reinforced

Triangle-SP graph



Remark: $\forall n, N_1(n) + N_2(n) = n$.

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Theorem (Mailler, V. 25+)

Almost surely,

$$\left(\frac{N_1(n)}{n}, \frac{N_2(n)}{n}, \frac{N_3(n)}{n}\right) \underset{n\to\infty}{\longrightarrow} w$$

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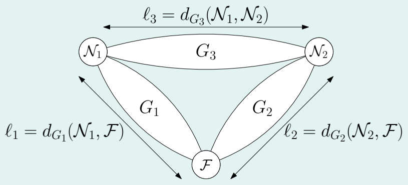

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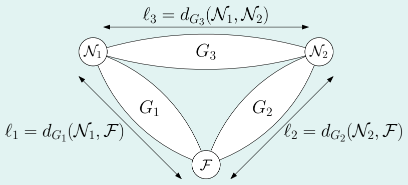

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$$\beta_1 = \frac{\alpha_1\ell_1(\ell_3+\ell_2-\ell_1)}{\ell_1\ell_3+(\ell_2-\ell_1)((1-\alpha_1)(\ell_3-\ell_2)+\alpha_1\ell_1)}$$

$$\beta_3 = \frac{\alpha_1\ell_3(1-\alpha_1)(\ell_1+\ell_2-\ell_3)}{(\ell_2-\ell_1)(\ell_1+\ell_2-\ell_3)\alpha_1+\ell_2(\ell_1-\ell_2+\ell_3)}$$

$= d_{G_2}(\mathcal{N}_2, \mathcal{F})$

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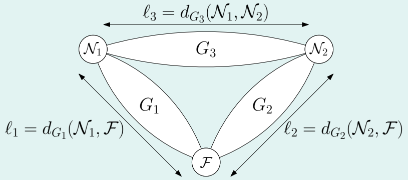

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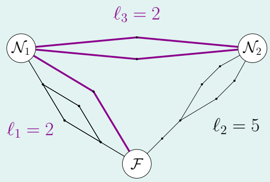

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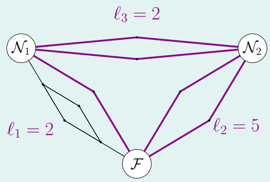

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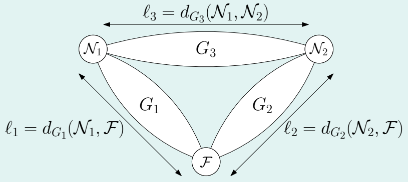
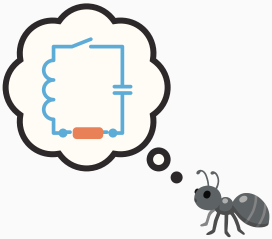


Toolbox:

- Pólya urns
- Stochastic approximation theory
- Conductance method and results on the one-nest model on SP-graphs

Toolbox & Proof

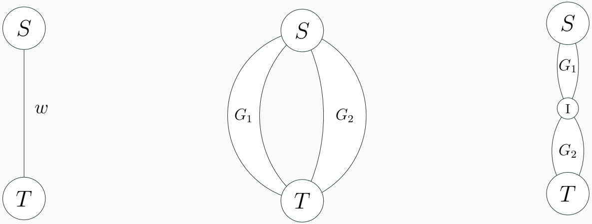
Conductance method



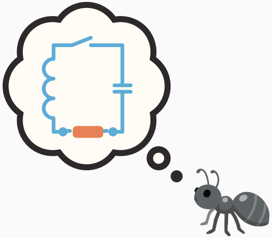
Conductance method

See *Probability on trees and networks [LP16]*

Effective conductance between two vertices - **recursive definition:**



(a) $C_G = w$ **(b)** $C_G = C_{G_1} + C_{G_2}$ **(c)** $C_G = \frac{1}{1/C_{G_1} + 1/C_{G_2}}$



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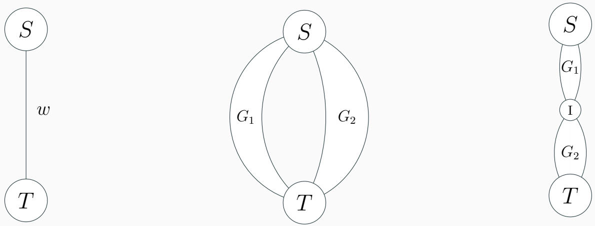

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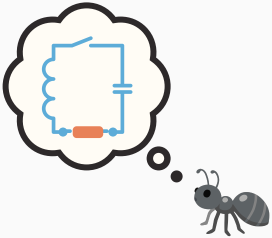
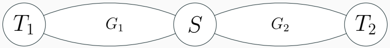


Key idea: the probability that a random walk starting from S hits T_1 before T_2 is $\frac{C_{G_1}}{C_{G_1} + C_{G_2}}$.



Conductances in the (LE) model on SP graphs

On SP graphs:

$$h_{\min}(G) = d(\mathcal{N}, \mathcal{F})$$

Theorem (Kious, Mailler, Schapira [KMS22a])

$$\frac{n}{h_{\max}(G)} \leq C_G(n) \leq \frac{n + C}{h_{\min}(G)}$$

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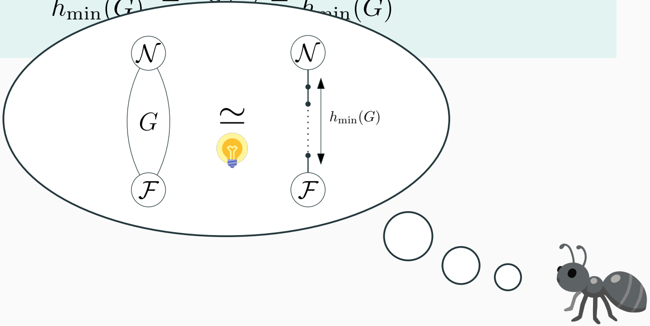
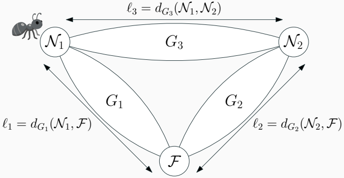


Preliminary computation



Example: If $\mathcal{N}(n) = \mathcal{N}_1$, the probability to reinforce in G_1 is

$$\frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}}$$

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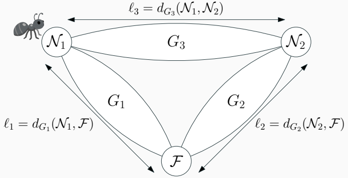

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Key to apply [KMS22a] results:

- conditionnal on $\gamma \in G_1$, γ is distributed as γ_1 obtained by doing a (LE) step in G_1 only
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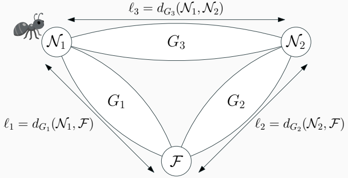

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Corollary

For every $i \in \{1, 2, 3\}$,

$$\frac{C_{G_i}(n)}{N_i(n)} \underset{n\to\infty}{\longrightarrow} \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

(with bounds for the convergence speed)

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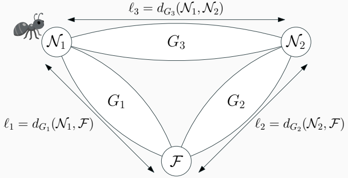

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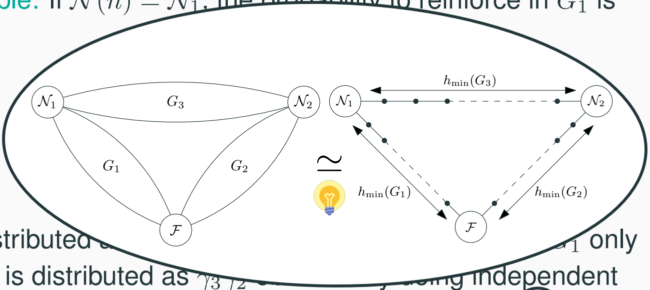

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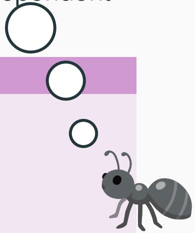

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Stochastic approximation

See *Random processes with reinforcement* [Pem07]

A process $(X_n)_{n \geq 0}$ is a **stochastic approximation** if

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- $F : \mathcal{E} \to \mathbb{R}^d$ is a Lipschitz function,

Stochastic approximation

See *Random processes with reinforcement* [Pem07]

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} = X_n + \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

and if

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- the remainder term r_n is $\mathcal{F}_n$-measurable and such that $\sum_n n^{-1}||r_n|| < \infty$ a.s.

Claim: the process $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n\geq 0}$ is a stochastic approximation !

Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right)$$

$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

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$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2 \left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$

Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

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For every $i \in \{1,2,3\}$,

$$\frac{C_{G_i}(n)}{N_i(n)} \longrightarrow \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

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Proof that the process is a stochastic approximation

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$$\sim \alpha_1 \frac{w_1/\ell_1}{w_1/\ell_1 + \frac{w_2/\ell_2\, w_3/\ell_3}{w_2/\ell_2+w_3/\ell_3}} + \alpha_2\left(1 - \frac{w_2/\ell_2}{w_2/\ell_2 + \frac{w_1/\ell_1\, w_3/\ell_3}{w_1/\ell_1+w_3/\ell_3}}\right) =: p_1(w_1, w_2, w_3)$$ (with $w_i = \hat{N}_i(n)$)

Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

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$$= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - p(\hat{N}(n)) + p(\hat{N}(n)) - \frac{N(n)}{n}\right)$$

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Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\begin{aligned}
\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} &= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right) \\
&= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - p(\hat{N}(n)) + p(\hat{N}(n)) - \frac{N(n)}{n}\right) \\
&= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}
\end{aligned}$$

And $\sum_n \frac{||r_n||}{n} < \infty$, because $\forall i \in \{1,2,3\}, \forall n, N_i(n) \geq n^{\varepsilon_i}$. $(*)$

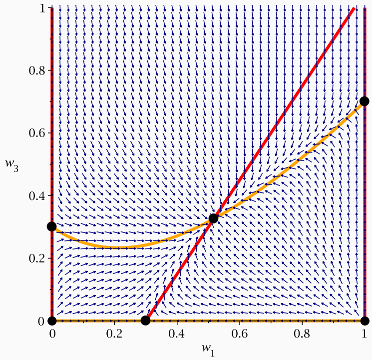
$$\begin{aligned}
\mathbb{E}[I|\hat{N}(n)]_1 &= \alpha_1 \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + \alpha_2\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right) \\
&\sim \alpha_1 \frac{w_1/\ell_1}{w_1/\ell_1 + \frac{w_2/\ell_2 \, w_3/\ell_3}{w_2/\ell_2 + w_3/\ell_3}} + \alpha_2\left(1 - \frac{w_2/\ell_2}{w_2/\ell_2 + \frac{w_1/\ell_1 \, w_3/\ell_3}{w_1/\ell_1 + w_3/\ell_3}}\right) =: p_1(w_1, w_2, w_3) \text{ (with } w_i = \hat{N}_i(n))
\end{aligned}$$

The ODE method

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} - X_n = \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

Vector field $F(w_1, w_3)$:



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

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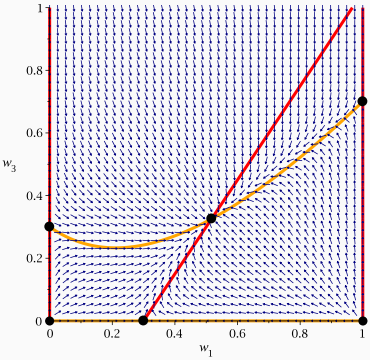

(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

ODE method

If there exists $p_1, \ldots, p_k$ s.t. for any $w \in [0,1]^2$, the solution of the ODE $\dot{y} = F(y)$ starting at w converges to some p_i, then almost surely,

$$\exists i : \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \xrightarrow[n\to\infty]{} p_i$$

Main idea: if ξ_{n+1} and r_n behave nicely, $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ *follows the flow of the ODE* $\dot{y} = F(y)$!

The ODE method

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} - X_n = \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

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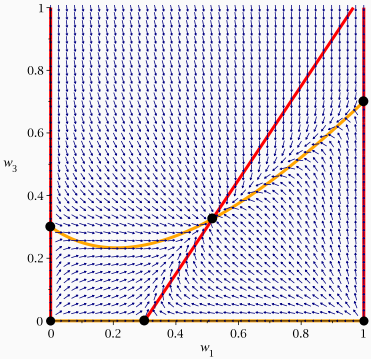
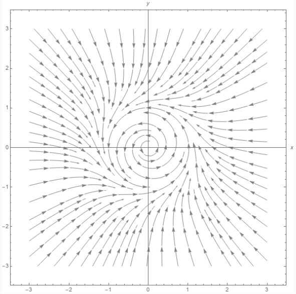


(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

What does not happen:

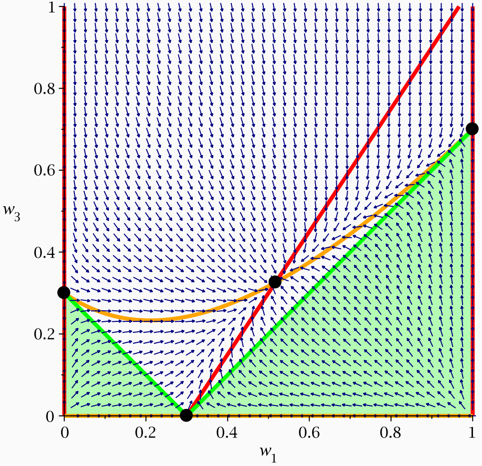


Conclusion: any solution to $\dot{y} = F(y)$ starting in $[0,1]^2$ converges

$\rightarrow \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ converges a.s. !

Eliminating the “bad” zeros

Vector field: $F(w_1, w_3)$



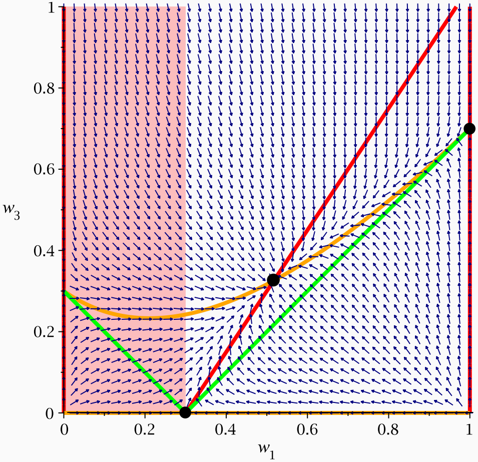
(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

$\liminf_{n\to\infty} \frac{N_1(n)+N_3(n)}{n} \geq \alpha$ and
$\liminf_{n\to\infty} \frac{N_2(n)+N_3(n)}{n} \geq 1-\alpha$

Lemma

Eliminating the "bad" zeros

Vector field: $F(w_1, w_3)$



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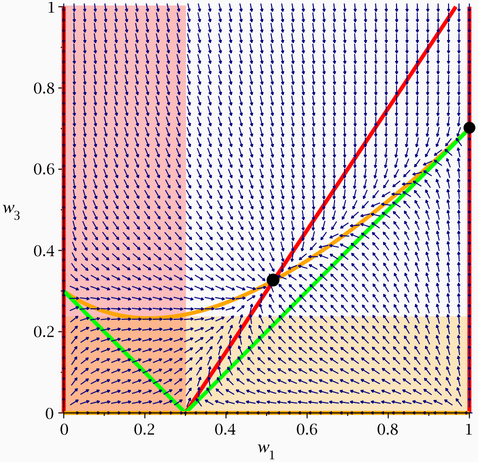
Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha$

Eliminating the “bad” zeros

Vector field: $F(w_1, w_3)$



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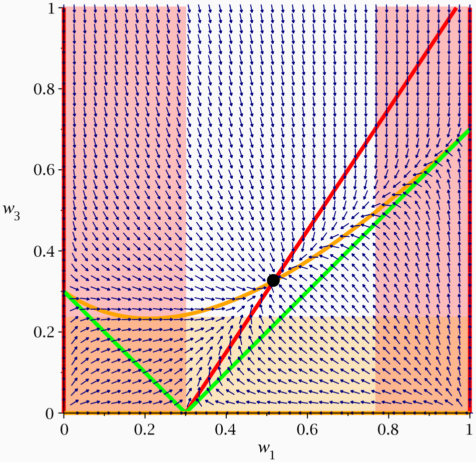
Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

Eliminating the “bad” zeros

Vector field: $F(w_1, w_3)$



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

$\liminf_{n\to\infty} \frac{N_1(n)+N_3(n)}{n} \geq \alpha$ and
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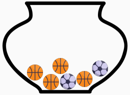
- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

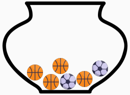
$N(n) := \#$🏀 at step n. In a classical Pólya urn:



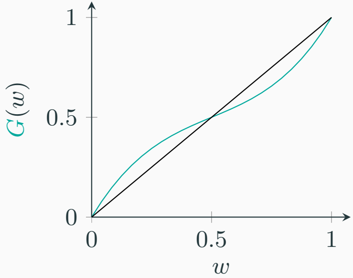
$$\mathbb{P}\left(N(n+1) = N(n) + 1 \,\middle|\, \frac{N(n)}{n} = w\right) = \quad w$$

Urn models

$N(n) :=$ #🏀 at step n. In a G-urn:



$$\mathbb{P}\left(N(n+1) = N(n) + 1 \,\middle|\, \frac{N(n)}{n} = w \right) = G(w)$$



Urn models

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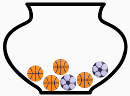
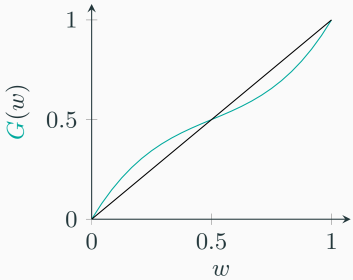


$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} W$, where W is a (random) stable fixed point of G.



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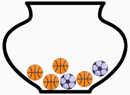

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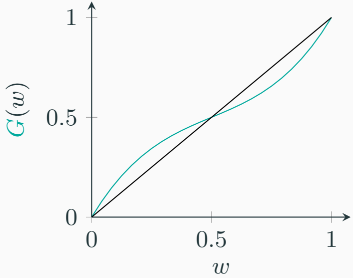
Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} W$, where W is a (random) stable fixed point of G.

Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0,1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$, $W = 0.5$ a.s.



Urn models

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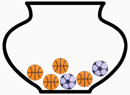

$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

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Almost surely, $\frac{N(n)}{n} \xrightarrow[n\to\infty]{} W$, where W is a (random) stable fixed point of G.

Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0,1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$, $W = 0.5$ a.s.

Use this on our two-dimensional process $(N_1(n), N_3(n))$

If, for any $w_3 \in [0,1]$,

$$\underbrace{\mathbb{P}\left(N_1(n+1) = N_1(n)+1 \,\middle|\, \frac{N_1(n)}{n} = w_1, \frac{N_3(n)}{n} = w_3\right)}_{\sim F(w_1,w_3)} \geq G(w_1)$$

and if every stable fixed point of G is larger than some c, then

$$\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq c$$

Conclusion in the different cases

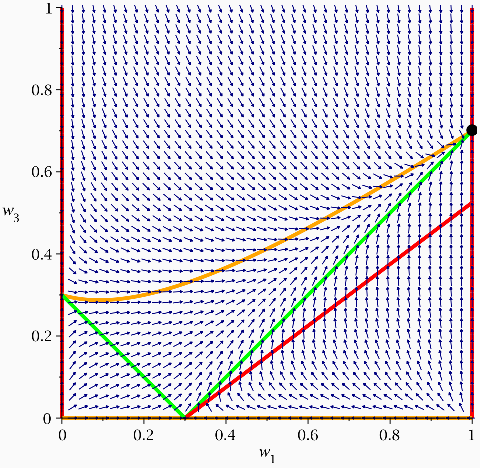


Figure 3: $\ell_1 = 2$, $\ell_2 = 6$ and $\ell_3 = 3$ (case $\ell_1 + \ell_3 < \ell_2$).
$\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (1, \alpha_2)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:
$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$
- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:
$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Conclusion in the different cases

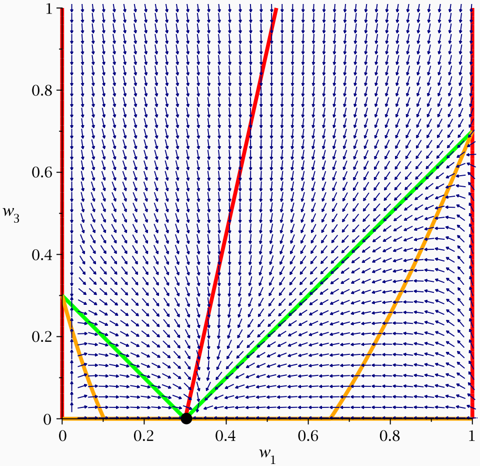


Figure 3: $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 9$ (case $\ell_1 + \ell_2 < \ell_3$).
$\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (\alpha_1, 0)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:
$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$
- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:
$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Conclusion in the different cases

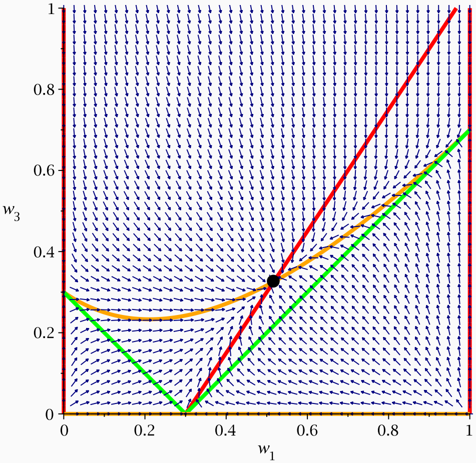
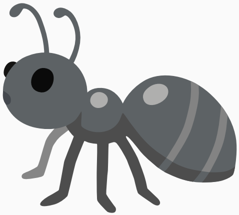


Figure 3: $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$ (case $\ell_1 + \ell_3 \geq \ell_2$ and $\ell_1 + \ell_2 \geq \ell_3$).
$\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (\beta_1, \beta_3)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:
$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$
- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:
$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Thank you !



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