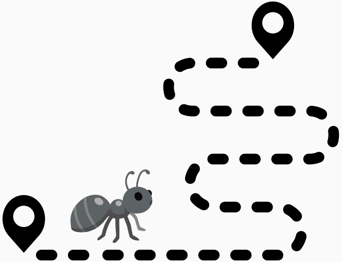
A probabilistic reinforcement-learning algorithm to find shortest paths in a graph

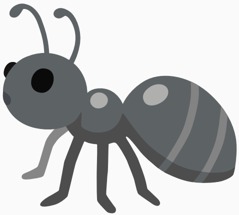
Zoé Varin (IRIF)

April 10th, 2026

Joint work with Cécile Mailler (University of Bath)



Introduction



Ants ?

Two biological ants' experiments:

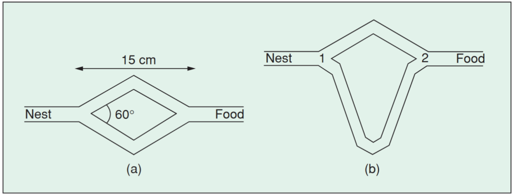


FIGURE 1 Experimental setup for the double bridge experiment.
(a) Branches have equal lengths [3]. (b) Branches have different lengths [4].

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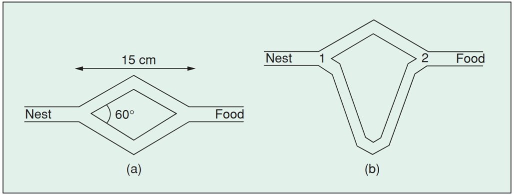

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Models to fit the experimental results:

$$p_1 = \frac{(m_1 + k)^h}{(m_1 + k)^h + (m_2 + k)^h}, \qquad (1)$$

(p_1 = probability to go in branch 1; m_i = quantity of pheromones in branch i; k and h parameters to fitted)

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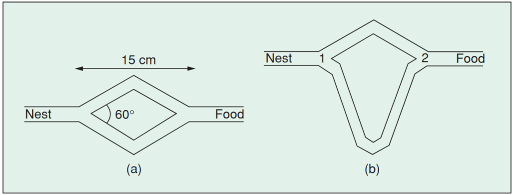

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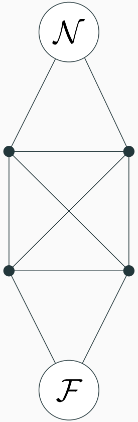
$\rightarrow$ **Definition of a Metaheuristic: Ant Colony Optimization**

See *Ant Colony Optimization* (Dorigo, Birattari and Stützle) for a nice introduction (and the source of the screenshots)

Definition of the ant model (one-nest version)



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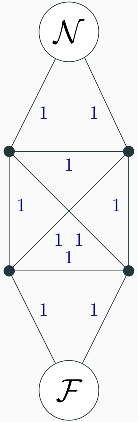

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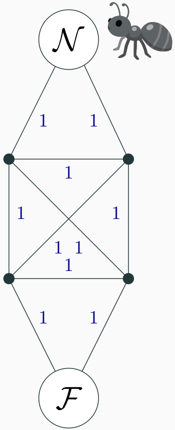

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starting from $\mathcal{N}$, stopped at $\mathcal{F}$

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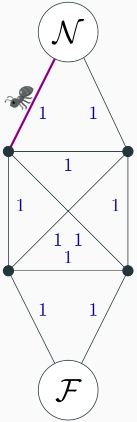

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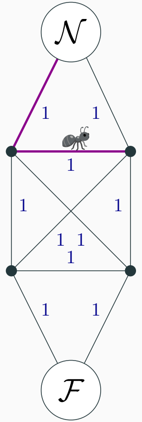

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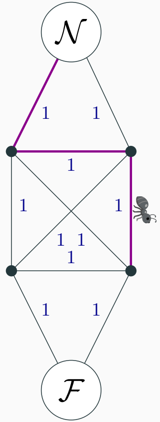

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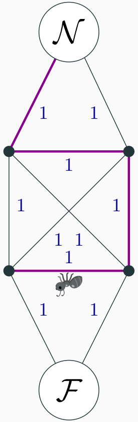

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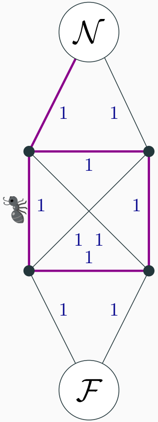

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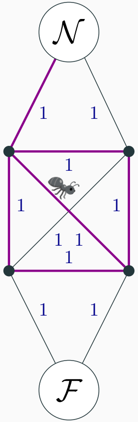

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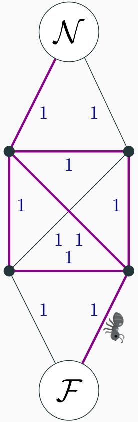

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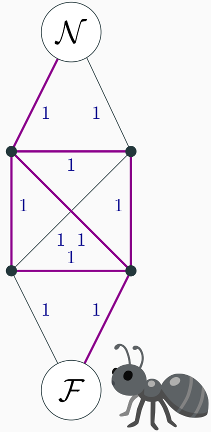

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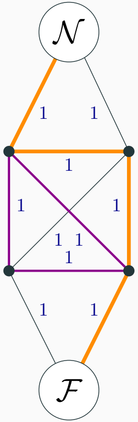

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Loop-erased (LE) model: $\gamma = LE(X)$

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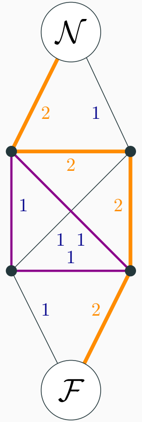

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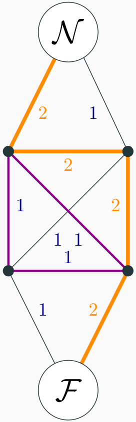

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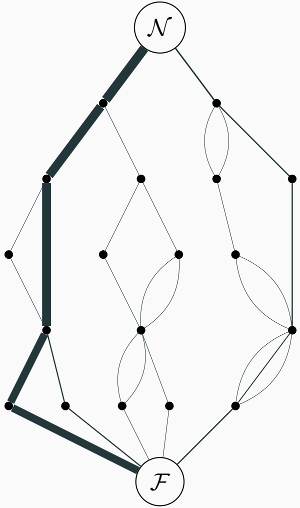

Simulations for $n = 10^8$

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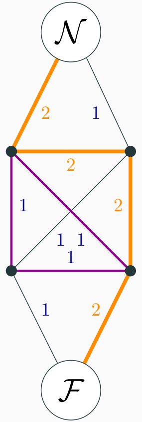

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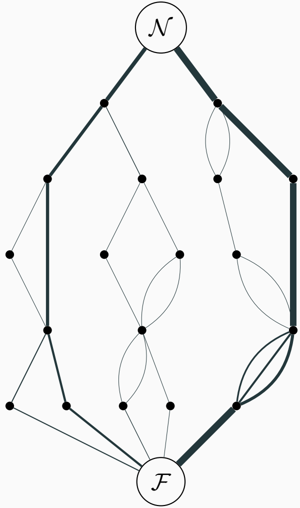

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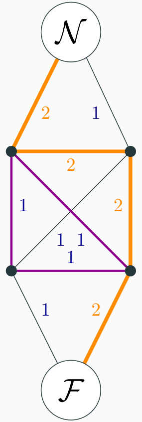

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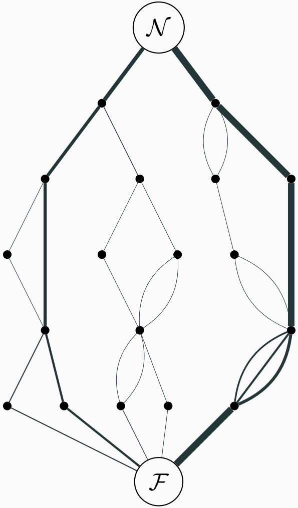
Question: Do the ants find shortest paths from $\mathcal{N}$ to $\mathcal{F}$?
$\to$ Does $\left(\frac{W_e(n)}{n}\right)_e$ converge ?
Towards which limit ?



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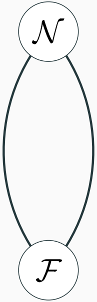
A quick warm-up



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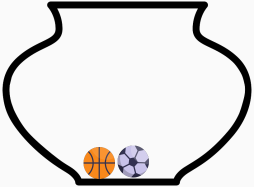

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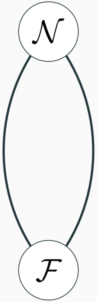

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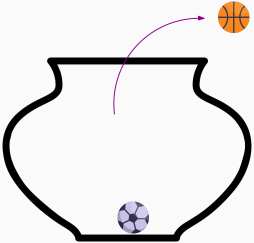

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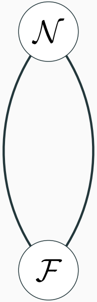

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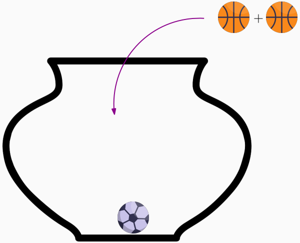

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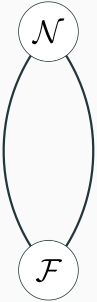

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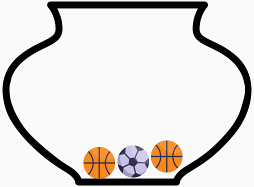

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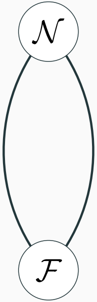

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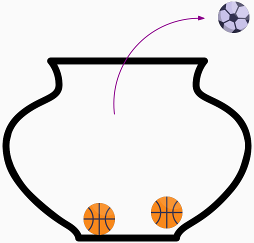

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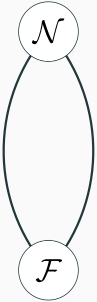

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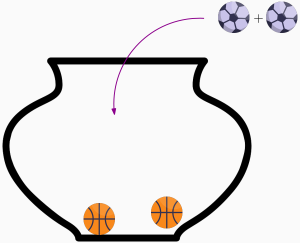

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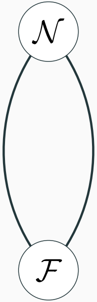

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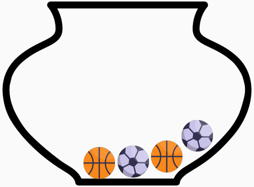

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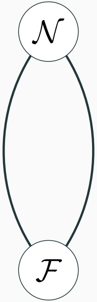

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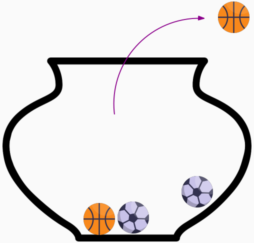

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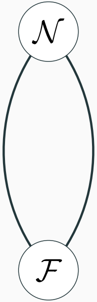

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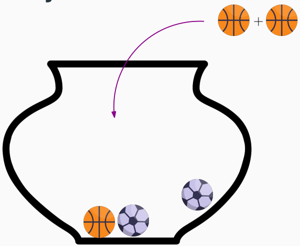

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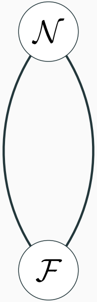

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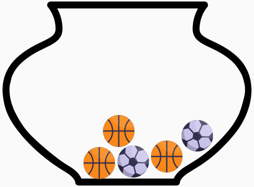

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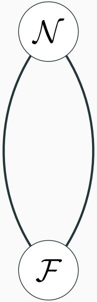

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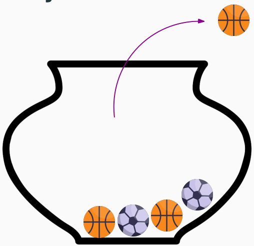

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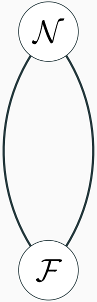

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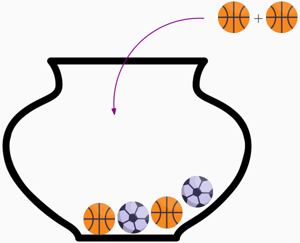

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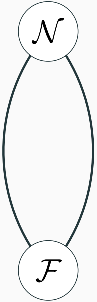

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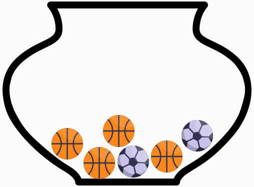

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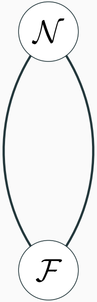

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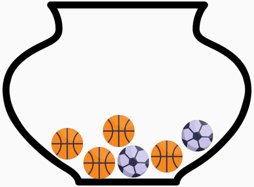
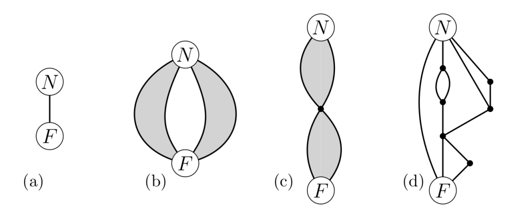


Asymptotic behavior:

Almost surely,

$$\frac{\#\text{🏀}}{n} \xrightarrow[n\to\infty]{} U \sim \mathcal{U}([0,1])$$

Recursive definition of series-parallel (SP) graphs:



Loop-erased (LE) model on series-parallel graphs

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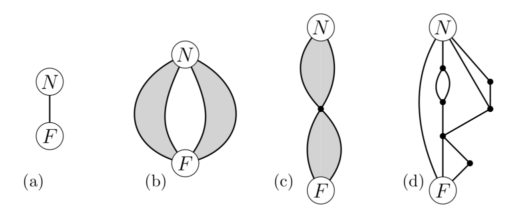

Theorem (Kious, Mailler, Schapira 22)

If G is a SP graph, then almost surely,

$$\frac{W_e(n)}{n} \underset{n\to\infty}{\longrightarrow} \chi_e, \quad \forall e \in E$$

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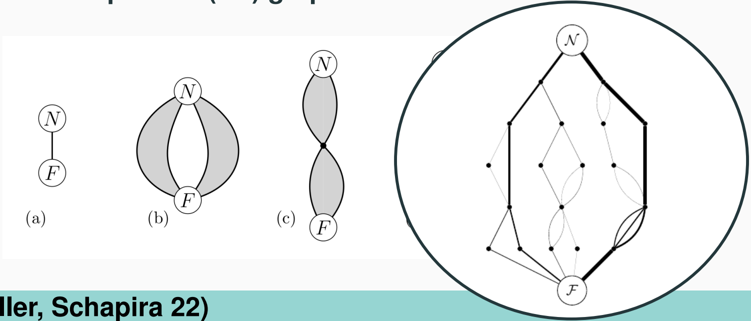
Loop-erased (LE) model on series-parallel graphs

Recursive definition of series-parallel (SP) graphs:



Theorem (Kious, Mailler, Schapira 22)

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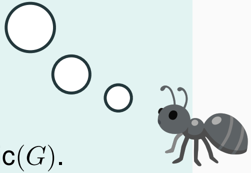
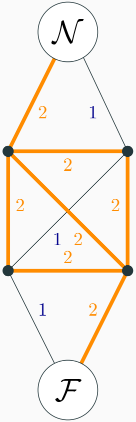
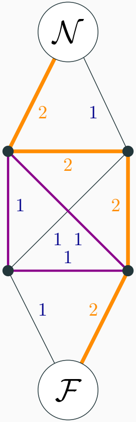


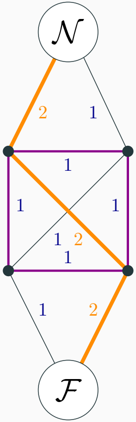
Other reinforcement models



(T) trace:
$\gamma = X$



(LE) loop-erased:
$\gamma = LE(X)$



(G) geodesic:
$\gamma = \text{ShortestPath}(X)$

At each step n:

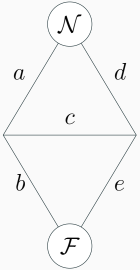
•**random walk** X:

•**depositing pheromones** on γ:

$$\forall e, W_e(n+1) = W_e(n) + \mathbb{1}_{e \in \gamma}$$

Geodesic (G) model on the lozenge graph

The lozenge graph:



Theorem (Kious, Mailler, Schapira 22)

Almost surely,

$$\frac{W_i(n)}{n} \underset{n\to\infty}{\longrightarrow} \chi_i, \quad \forall i \in \{a,b,c,d,e\}$$

where $(\chi_i)_{i\in\{a,b,c,d,e\}}$ is a random vector, such that almost surely, $\chi_c = 0$ and $\chi_a = \chi_b = 1 - \chi_d = 1 - \chi_e \in (0,1)$.

Conjecture for the loop-erased (LE) and geodesic (G) models

Conjecture (KMS22)

Almost surely,

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where $(\chi_e)_{e\in E}$ is a random vector such that

- (LE) model $\chi_e \neq 0$ a.s. **if and only if** e belongs to a shortest path from N to F
- (G) model $\chi_e \neq 0$ a.s. **only if** e belongs to a shortest path from N to F

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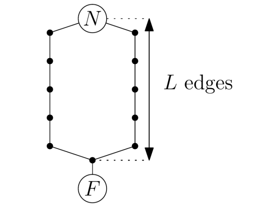
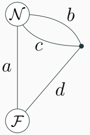


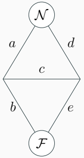
For L large enough, there exists e such that

$$\mathbb{P}\left(W_e(n)/n \to 0\right) > 0$$

Trace (T) model (Results from KMS22)

Some examples: the cone and the lozenge



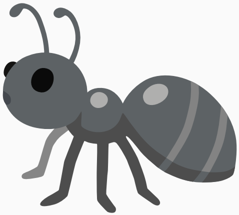


$$\frac{W(n)}{n} \xrightarrow[n\to\infty]{} (1, 1/3, 1/3, 0) \qquad \frac{W(n)}{n} \xrightarrow[n\to\infty]{} (w^\star, 1/2, 1/2, w^\star, 1/2)$$

Theorem: convergence for a family of *tree-like* graphs (s.t. $G\backslash\{\mathcal{F}\}$ is a tree).

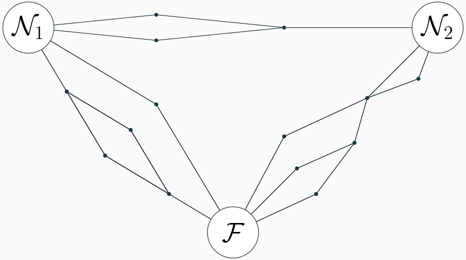
Conjecture: deterministic limit for any graph without multiple-edges adjacent to F.

The two-nest model



Back to the (LE) model: multinest version

2-nest version: at every step n, $\mathcal{N}(n) = \begin{cases} \mathcal{N}_1 & \text{with proba } \alpha \in (0,1) \\ \mathcal{N}_2 & \text{with proba } 1-\alpha \end{cases}$.



Back to the (LE) model: multinest version on triangle-SP graphs

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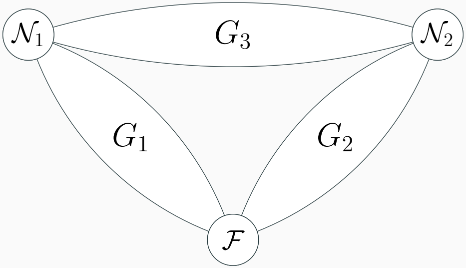
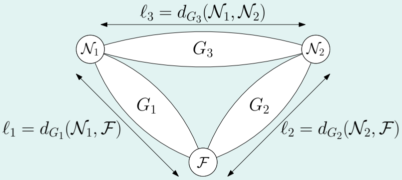


Triangle-SP graph: G_1, G_2, G_3 series-parallel graphs

Our main result: convergence of the 2-nest LE model on triangle-SP graphs

$N_i(n)$ = number of steps at which edges in G_i have been reinforced

Triangle-SP graph



Remark: $\forall n, N_1(n) + N_2(n) = n$.

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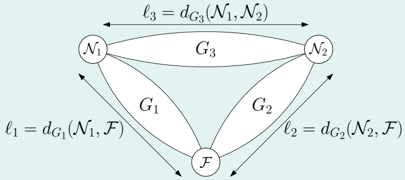
Theorem (Mailler, V. 25+)

Almost surely,

$$\left(\frac{N_1(n)}{n}, \frac{N_2(n)}{n}, \frac{N_3(n)}{n}\right) \underset{n\to\infty}{\longrightarrow} w$$

Triangle-SP graph



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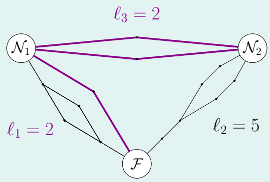

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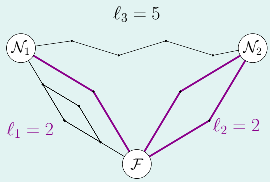

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$$\beta_1 = \frac{\alpha_1\ell_1(\ell_3+\ell_2-\ell_1)}{\ell_1\ell_3+(\ell_2-\ell_1)((1-\alpha_1)(\ell_3-\ell_2)+\alpha_1\ell_1)}$$

$$\beta_3 = \frac{\alpha_1\ell_3(1-\alpha_1)(\ell_1+\ell_2-\ell_3)}{(\ell_2-\ell_1)(\ell_1+\ell_2-\ell_3)\alpha_1+\ell_2(\ell_1-\ell_2+\ell_3)}$$

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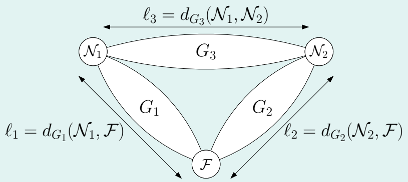

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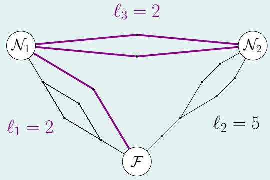

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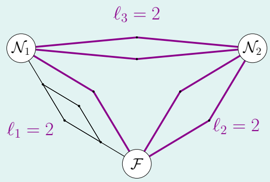

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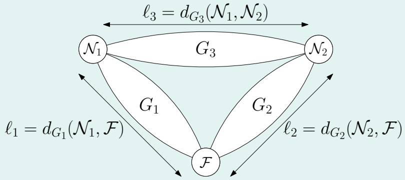
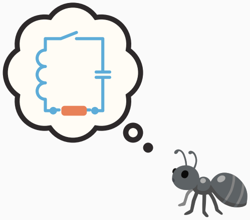


Toolbox:

- Pólya urns
- Stochastic approximation theory
- Conductance method and results on the one-nest model on SP-graphs

Toolbox & Proof

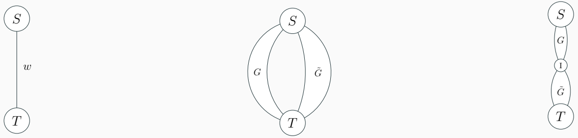
Conductance method



Conductance method

See *Probability on trees and networks* (Lyons, Peres 16)

Effective conductance between two vertices - **recursive definition for SP graphs:**



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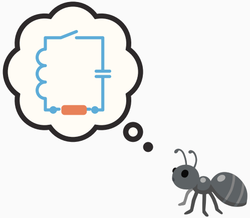

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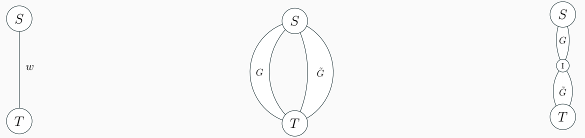

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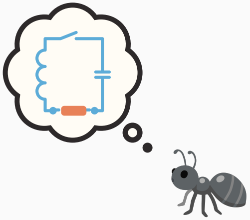
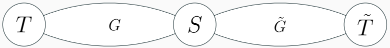


Key idea: the probability that a random walk starting from S hits T before $\tilde{T}$ is $\frac{C_G}{C_G + C_{\tilde{G}}}$.



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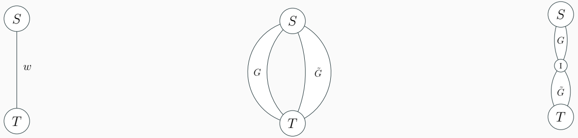

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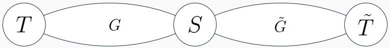

Example: If the ant starts from $\mathcal{N}_1$, the probability to reinforce in G_1 is

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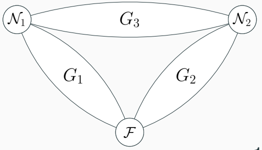

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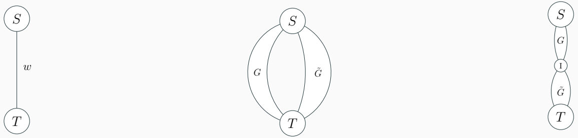

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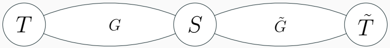

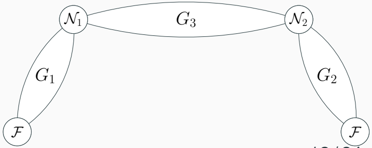

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On SP graphs:

$h_{\min}(G) = d_G(\mathcal{N}, \mathcal{F})$

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(with bounds for the convergence speed)

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Combinatorial analysis of LE walks

- conditional on $\gamma \in G_1$, γ is distributed as γ_1 obtained by doing a (LE) step in G_1 only
- conditional on $\gamma \in G_3 \cup G_2$, γ is distributed as $\gamma_3\gamma_2$ obtained by doing independent (LE) steps in G_3 and G_2 only.

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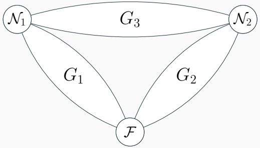
- conditional on $\gamma \in G_3 \cup G_2$, γ is distributed as $\gamma_3\gamma_2$ obtained by doing independent (LE) steps in G_3 and G_2 only.

Two-nest model:



Proposition (Mailler, V. 25+)

For every $i \in \{1, 2, 3\}$,

$$\frac{C_{G_i}(n)}{N_i(n)} \underset{n\to\infty}{\longrightarrow} \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

(Partial) summary of the proof

Step 1: Conductance method. $\mathbb{P}\left(N_1(n+1) - N_1(n) = 1\right) =$ Function$(C_{G_1}, C_{G_2}, C_{G_3})$.

Step 2: $C_{G_i}(n) \underset{n\to\infty}{\sim} \frac{N_i(n)}{\ell_i}$. (KMS22 + combinatorial analysis of the walks)

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Step 3: Stochastic approximation. The error made in Step 2 is asymptotically negligible. Thus $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n\geq 0}$ is a stochastic approximation + converges.

Stochastic approximation

See *Random processes with reinforcement* [Pem07]

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} = X_n + \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

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and if

- $(X_n)_{n \geq 0}$ is adapted to some filtration $(\mathcal{F}_n)_{n \geq 0}$, and takes value in some convex compact $\mathcal{E} \subseteq \mathbb{R}^d$,

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- the noise ξ_{n+1} is $\mathcal{F}_{n+1}$-measurable and such that $\forall n, \mathbb{E}_n\left[\xi_{n+1}\right] = 0$,
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- the remainder term r_n is $\mathcal{F}_n$-measurable and such that $\sum_n n^{-1}||r_n|| < \infty$ a.s.

Claim: the process $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n\geq 0}$ is a stochastic approximation !

Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right)$$

$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

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$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + (1-\alpha)\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$

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For every $i \in \{1,2,3\}$,

$$C_{G_i}(n) \sim \frac{N_i(n)}{\ell_i}$$

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$$\sim \alpha \frac{w_1/\ell_1}{w_1/\ell_1 + \frac{w_2/\ell_2 \, w_3/\ell_3}{w_2/\ell_2 + w_3/\ell_3}} + (1-\alpha)\left(1 - \frac{w_2/\ell_2}{w_2/\ell_2 + \frac{w_1/\ell_1 \, w_3/\ell_3}{w_1/\ell_1 + w_3/\ell_3}}\right) =: p_1(w_1, w_2, w_3)$$ (with $w_i = \hat{N}_i(n)$)

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We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

$$\begin{aligned}
\frac{N(n+1)}{n+1} = \frac{N(n)+I}{n+1} &= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n}\right) \\
&= \frac{N(n)}{n} + \frac{1}{n+1}\left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - p(\hat{N}(n)) + p(\hat{N}(n)) - \frac{N(n)}{n}\right) \\
&= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[I|\hat{N}(n)]_1 &= \alpha \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + (1-\alpha)\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right) \\
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We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ and $I = \left(\mathbb{1}_{N_i(n+1)=N_i(n)+1}\right)_{i=1,3}$

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$$= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1}$$

And $\sum_n \frac{||r_n||}{n} < \infty$, because $\forall i \in \{1,2,3\}, \forall n, N_i(n) \geq n^{\varepsilon_i}$. $(*)$

$$\mathbb{E}[I|\hat{N}(n)]_1 = \alpha \frac{C_{G_1}(n)}{C_{G_1}(n) + \frac{C_{G_2}(n)C_{G_3}(n)}{C_{G_2}(n)+C_{G_3}(n)}} + (1-\alpha)\left(1 - \frac{C_{G_2}(n)}{C_{G_2}(n) + \frac{C_{G_1}(n)C_{G_3}(n)}{C_{G_1}(n)+C_{G_3}(n)}}\right)$$
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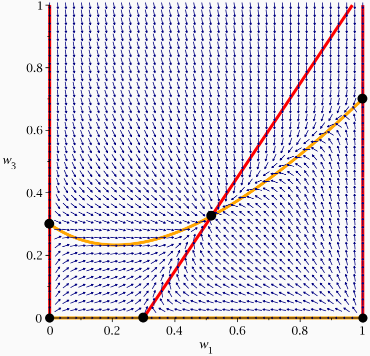
The ODE method

A process $(X_n)_{n\geq 0}$ is a **stochastic approximation** if

$$X_{n+1} - X_n = \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \ \forall n$$

Vector field $F(w_1, w_3)$:



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

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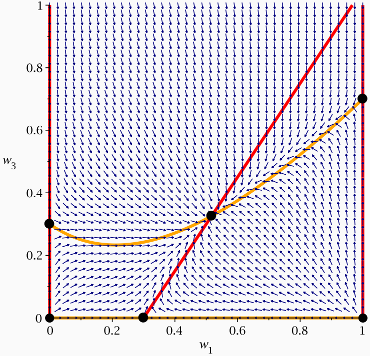

(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

Main idea: if ξ_{n+1} and r_n behave nicely, $(X_n)_{n\leq 0}$ *follows the flow of the ODE* $\dot{y} = F(y)$!

ODE method

If any solution of the ODE $\dot{y} = F(y)$ converges, then almost surely,

$$\exists \bullet : X_n \xrightarrow[n\to\infty]{} \bullet$$

where $\bullet$ is a (random) 0 of F.

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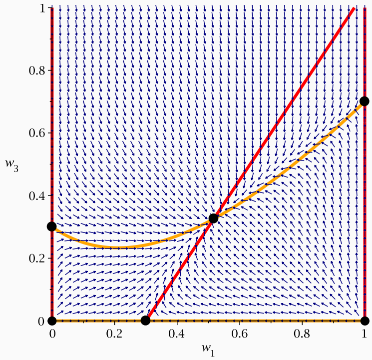
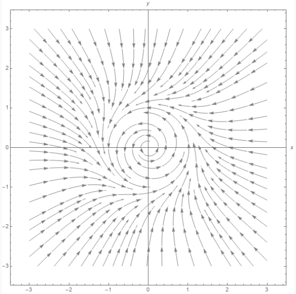


(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

What does not happen:



Conclusion: any solution to $\dot{y} = F(y)$ starting in $[0,1]^2$ converges

$\rightarrow \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)$ converges a.s. !

Conclusion of the proof

Step 1: Conductance method. $\mathbb{P}(N_1(n+1) - N_1(n) = 1) =$ Function$(C_{G_1}, C_{G_2}, C_{G_3})$.

Step 2: $C_{G_i}(n) \underset{n\to\infty}{\sim} \frac{N_i(n)}{\ell_i}$. (KMS22 + combinatorial analysis of the walks)

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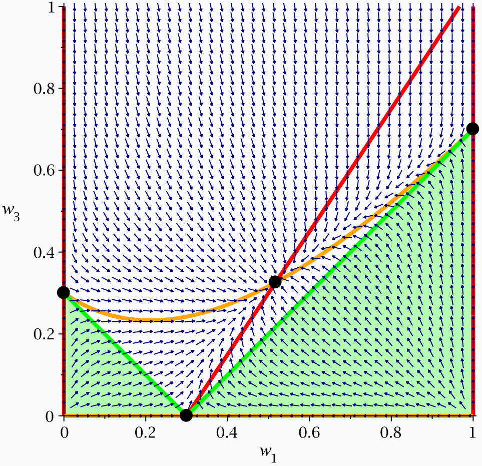
Step 4: Comparison with generalized Pólya urns to determine the limit.

Eliminating the "bad" zeros

Vector field: $F(w_1, w_3)$



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

$\liminf_{n\to\infty} \frac{N_1(n)+N_3(n)}{n} \geq \alpha$ and
$\liminf_{n\to\infty} \frac{N_2(n)+N_3(n)}{n} \geq 1-\alpha$

Lemma

Eliminating the “bad” zeros

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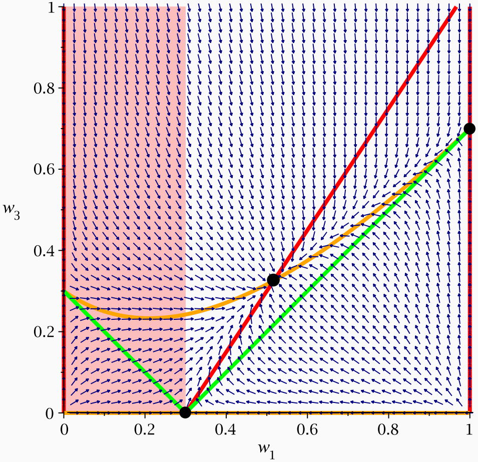

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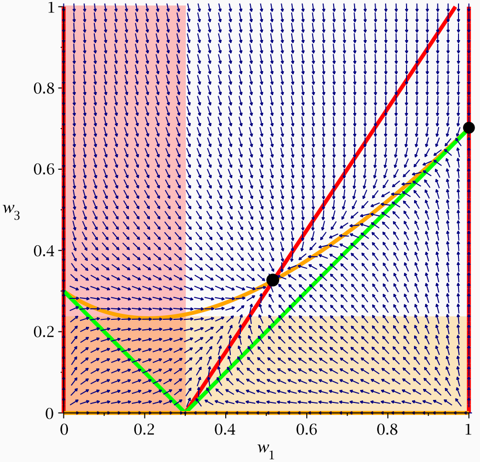

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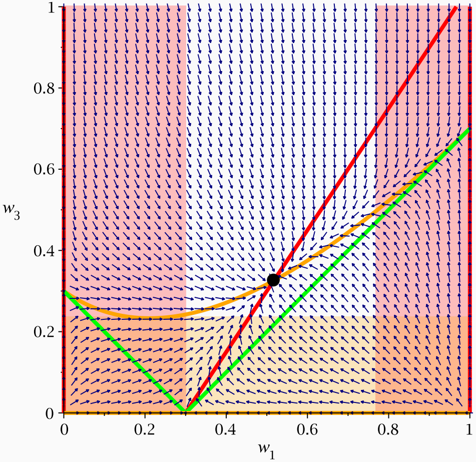
Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

Eliminating the "bad" zeros

Vector field: $F(w_1, w_3)$



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

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Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

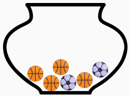
$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Urn models

$N(n) := \#$🏀 at step n. In a classical Pólya urn:



$$\mathbb{P}\left(N(n+1) = N(n) + 1 \,\middle|\, \frac{N(n)}{n} = w\right) = \quad w$$

Urn models

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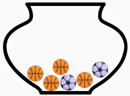

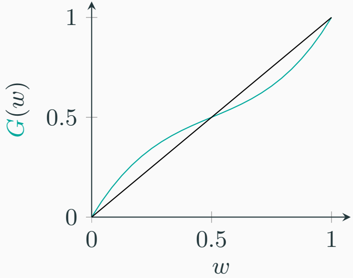

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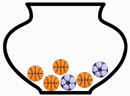
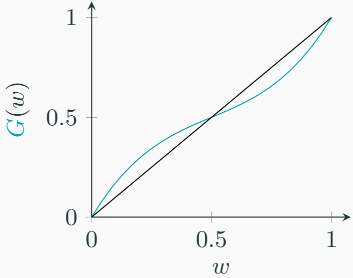


$$\mathbb{P}\left(N(n+1) = N(n) + 1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} W$, where W is a (random) stable fixed point of G.



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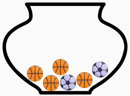

$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

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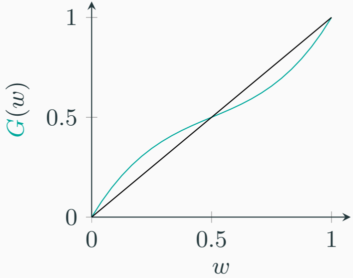
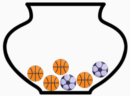


Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0,1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$, $W = 0.5$ a.s.

Urn models

$N(n) :=$ #🏀 at step n. In a G-urn:



$$\mathbb{P}\left(N(n+1) = N(n)+1 \,\middle|\, \frac{N(n)}{n} = w\right) = G(w)$$

w is a **stable fixed point** if $G(w) = w$ and $G'(w) \leq 1$

Convergence of G-urn processes

Almost surely, $\frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} W$, where W is a (random) stable fixed point of G.

Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0,1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$, $W = 0.5$ a.s.

💡 Use this on our two-dimensional process $(N_1(n), N_3(n))$

If, for any $w_3 \in [0,1]$,

$$\underbrace{\mathbb{P}\left(N_1(n+1) = N_1(n)+1 \,\middle|\, \frac{N_1(n)}{n} = w_1, \frac{N_3(n)}{n} = w_3\right)}_{\sim F(w_1,w_3)} \geq G(w_1)$$

and if every stable fixed point of G is larger than some c, then

$$\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq c$$

Conclusion in the different cases

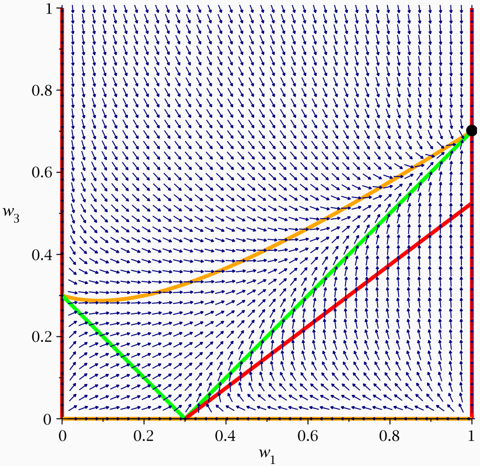


Figure 3: $\ell_1 = 2$, $\ell_2 = 6$ and $\ell_3 = 3$ (case $\ell_1 + \ell_3 < \ell_2$).
$\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (1, \alpha_2)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:
$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$
- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:
$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Conclusion in the different cases

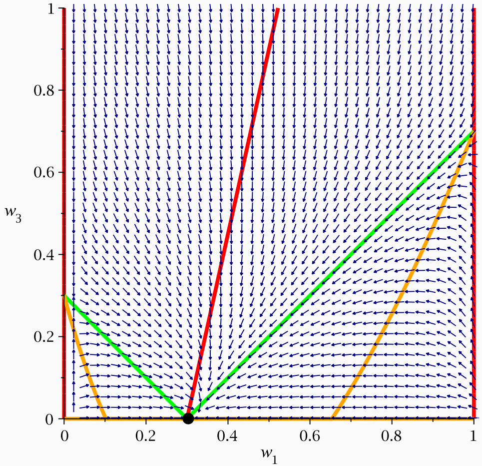


Figure 3: $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 9$ (case $\ell_1 + \ell_2 < \ell_3$). $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (\alpha_1, 0)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

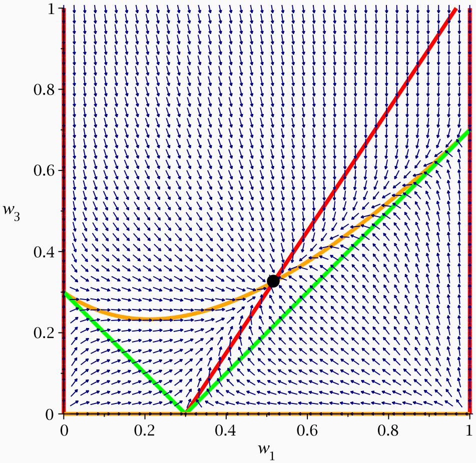


Figure 3: $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$ (case $\ell_1 + \ell_3 \geq \ell_2$ and $\ell_1 + \ell_2 \geq \ell_3$). $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \to (\beta_1, \beta_3)$

Lemma

- $\liminf_{n\to\infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

$$\liminf_{n\to\infty} \frac{N_3(n)}{n} \geq c$$

- if $\ell_2 < \ell_1 + \ell_3$, $\exists c' < 1$:

$$\limsup_{n\to\infty} \frac{N_1(n)}{n} \leq c'$$

Conclusion of the proof

Step 1: Conductance method. $\mathbb{P}\left(N_1(n+1) - N_1(n) = 1\right) = \text{Function}(C_{G_1}, C_{G_2}, C_{G_3})$.

Step 2: $C_{G_i}(n) \underset{n\to\infty}{\sim} \frac{N_i(n)}{\ell_i}$. (KMS22 + combinatorial analysis of the walks)

Step 3: Stochastic approximation. The error made in Step 2 is asymptotically negligible. Thus $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n\geq 0}$ is a stochastic approximation + converges.

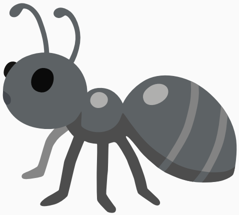
Step 4: Comparison with generalized Pólya urns to determine the limit.

Perspectives

A lot of unanswered questions remain

- Can we characterize the distribution of the edge weights' limit ?
- What happens on other families of graphs ?
- Or if we consider graphs with more nests ?
- Evaporation of the pheromones ?

Thank you !



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