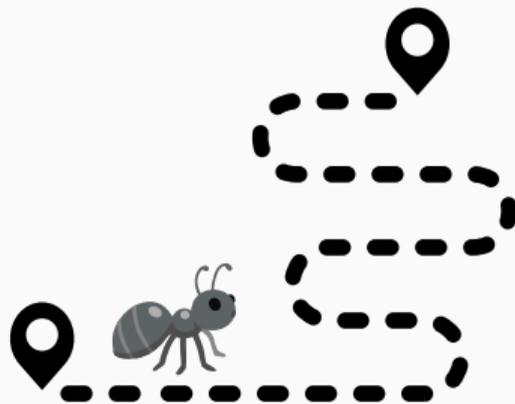


A probabilistic reinforcement-learning algorithm to find shortest paths in a graph

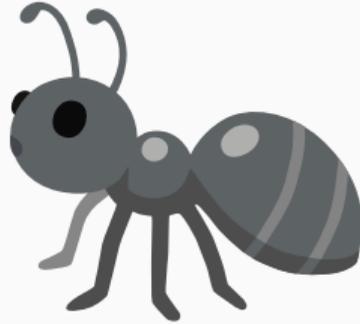
Zoé Varin (IRIF)

February 11th, 2026

Joint work with Cécile Mailler (University of Bath)



Introduction



Two biological ants' experiments:

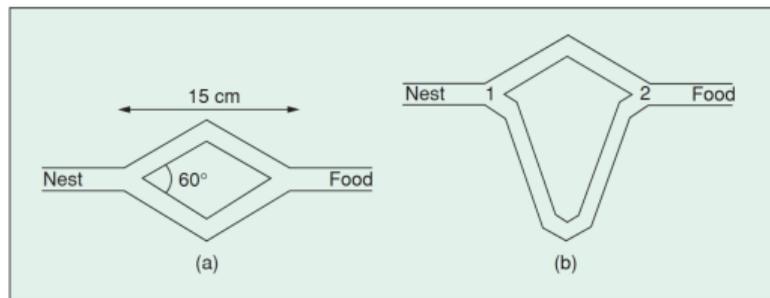


FIGURE 1 Experimental setup for the double bridge experiment.
(a) Branches have equal lengths [3]. (b) Branches have different lengths [4].

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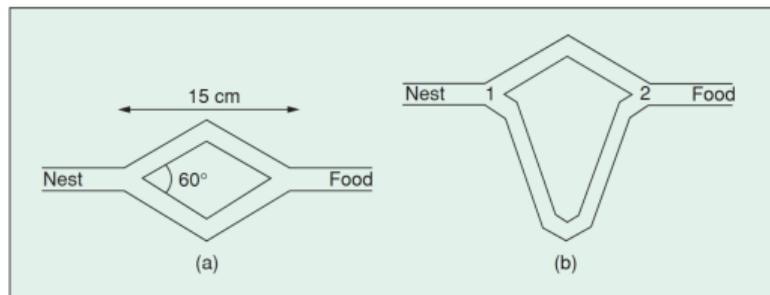


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Models to fit the experimental results:

$$p_1 = \frac{(m_1 + k)^h}{(m_1 + k)^h + (m_2 + k)^h}, \quad (1)$$

(p_1 = probability to go in branch 1; m_i = quantity of pheromones in branch i ; k and h parameters to fitted)

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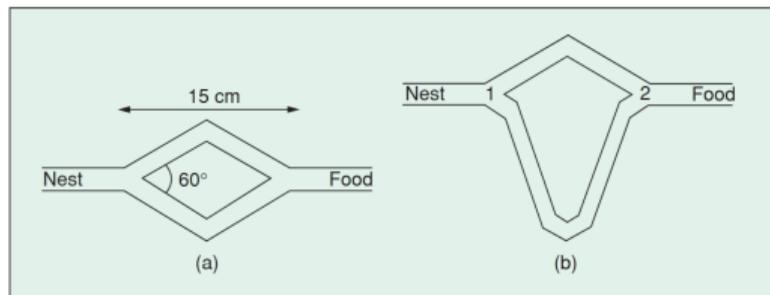


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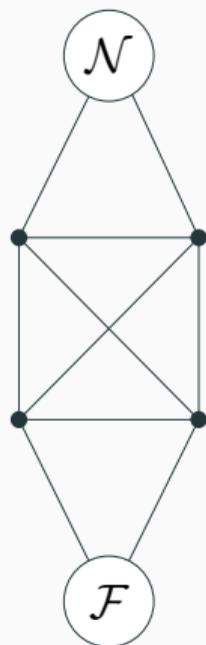
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→ **Definition of a Metaheuristic: Ant Colony Optimization**

📖 See *Ant Colony Optimization* (Dorigo, Birattari and Stützle) for a nice introduction (and the source of the screenshots)

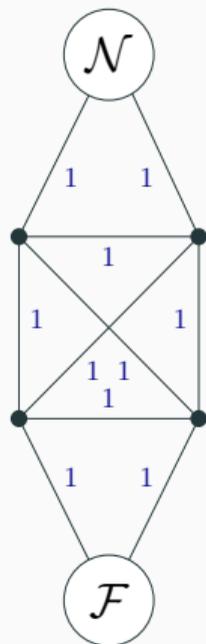
Definition of the ant model (one-nest version)

At each step n :

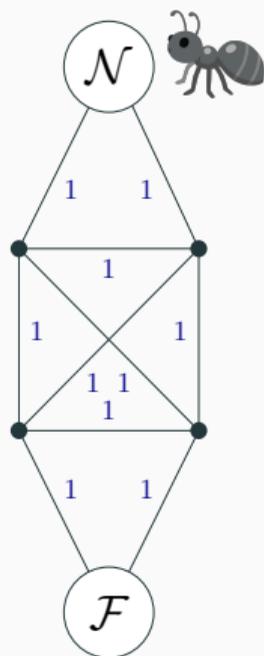


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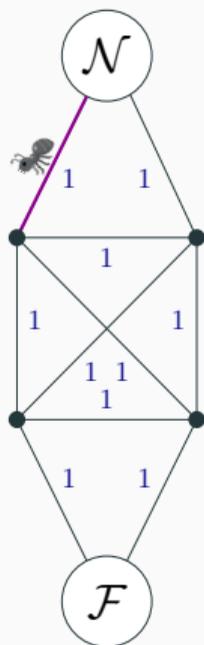
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- **random walk** X weighted by $W(n)$:

$$\mathbb{P}(u \rightarrow v) = \frac{W_{uv}(n)}{\sum_{e:u \in e} W_e(n)}$$

starting from \mathcal{N} , stopped at \mathcal{F}

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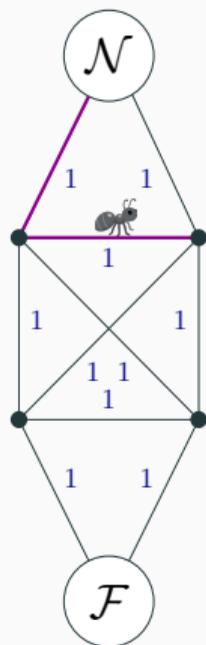
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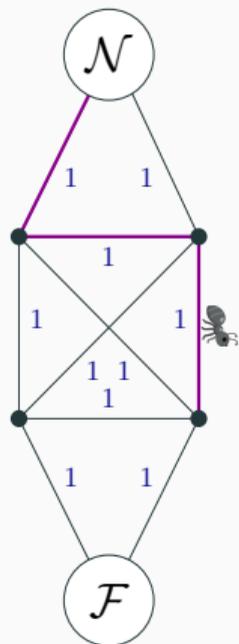
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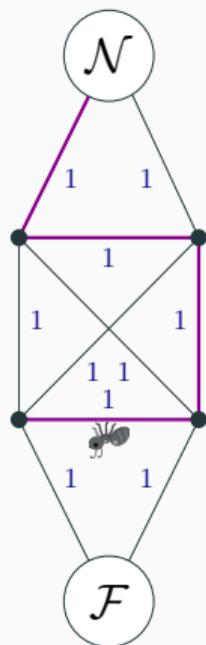
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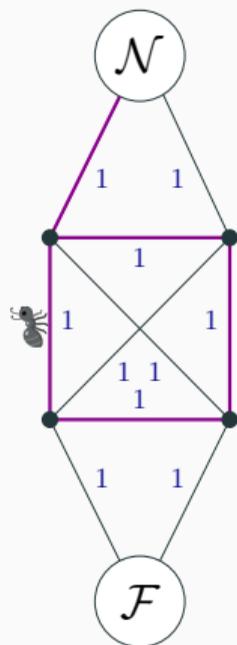
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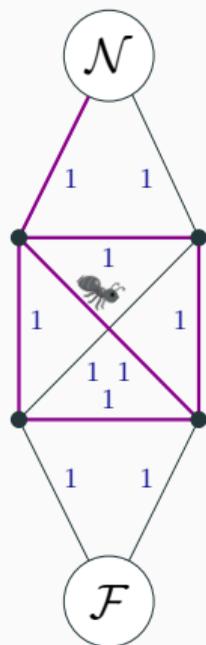
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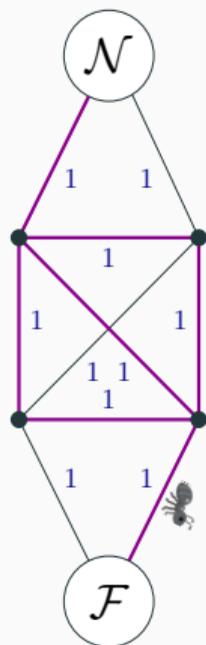
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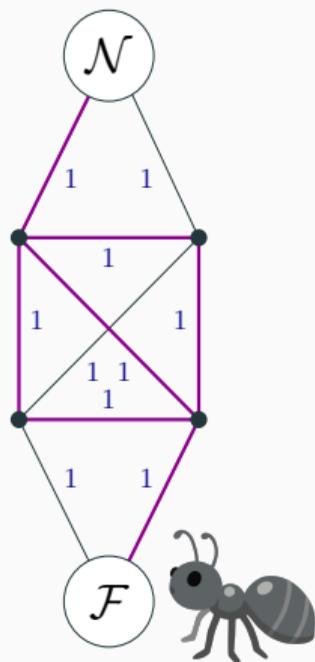
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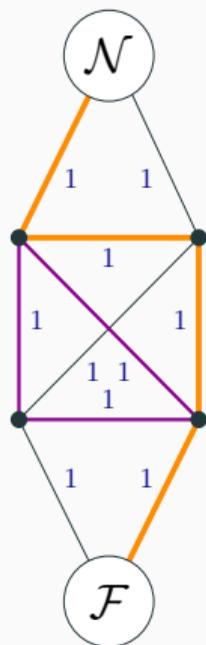
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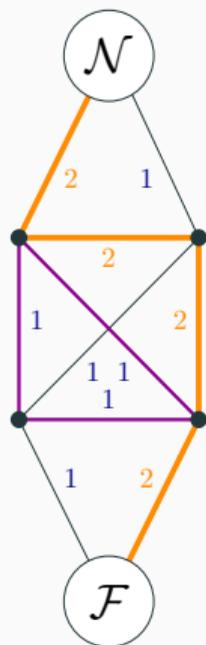
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- **depositing pheromones** on γ on their way back:

$$\forall e, W_e(n+1) = W_e(n) + \mathbb{1}_{e \in \gamma}$$

Loop-erased (LE) model: $\gamma = LE(X)$

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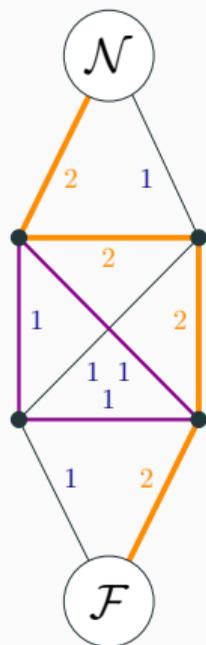
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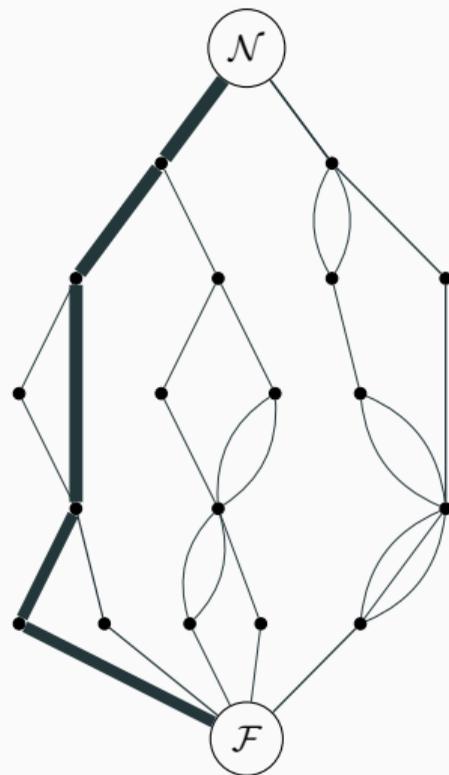
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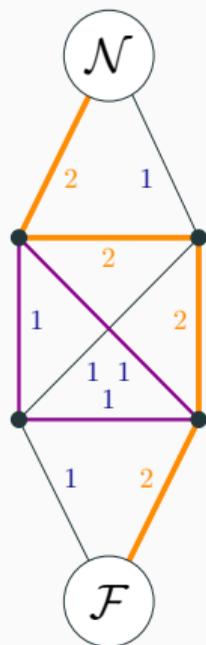
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Simulations for $n = 10^8$

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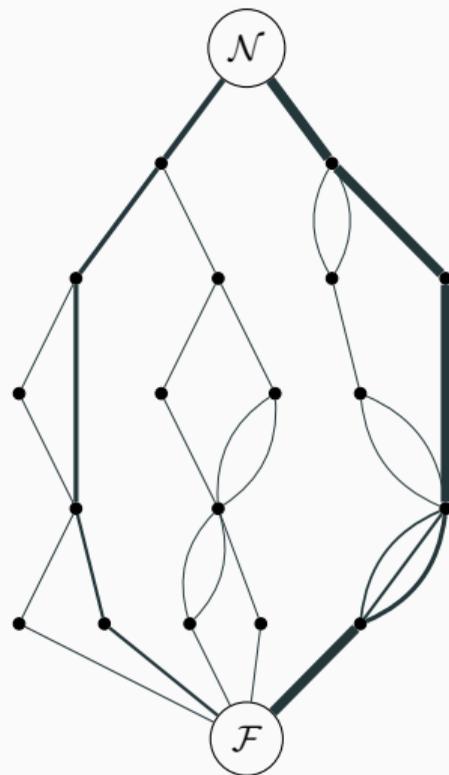
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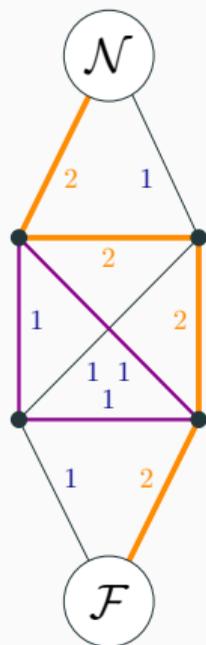
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Question: Do the ants find shortest paths from \mathcal{N} to \mathcal{F} ?

→ Does $\left(\frac{W_e(n)}{n}\right)_e$ converge ?

Towards which limit ?

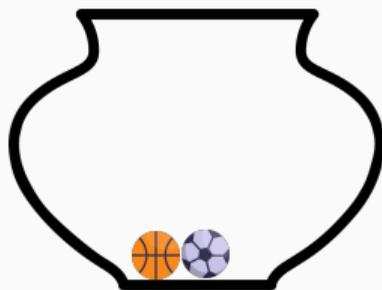


Simulations for $n = 10^8$

A quick warm-up



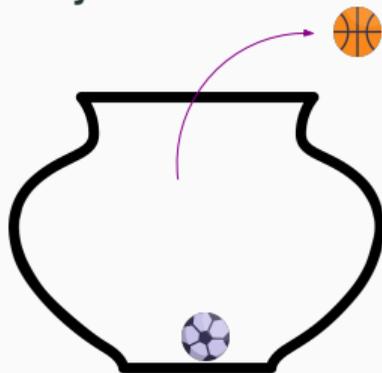
Pólya's urn:



A quick warm-up



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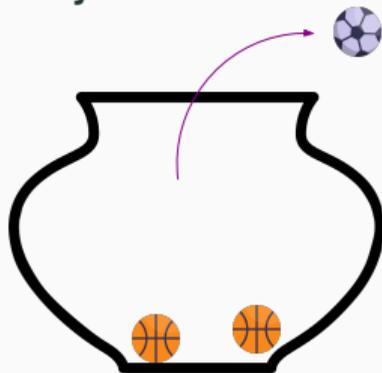
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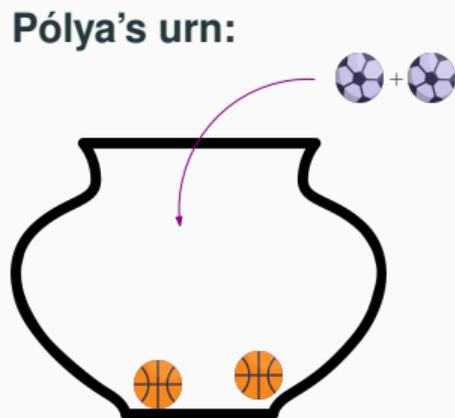
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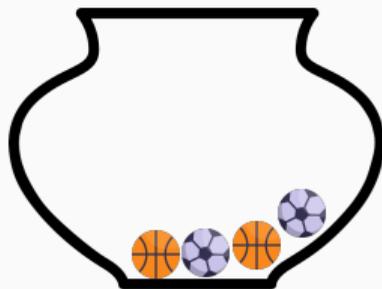
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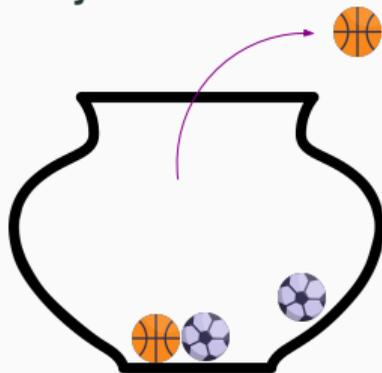
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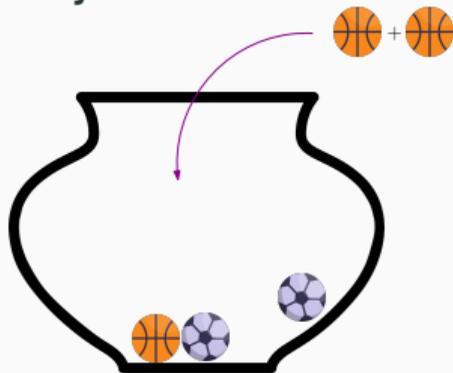
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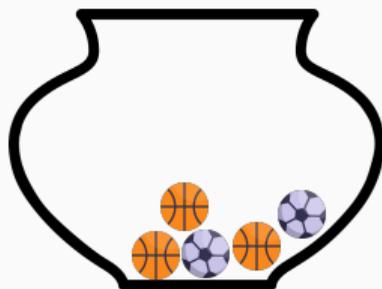
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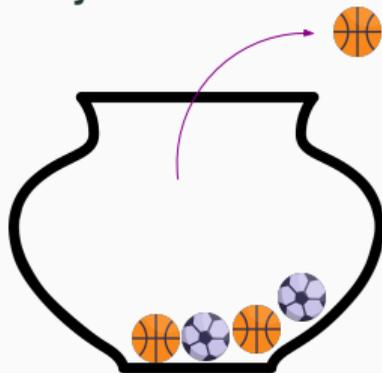
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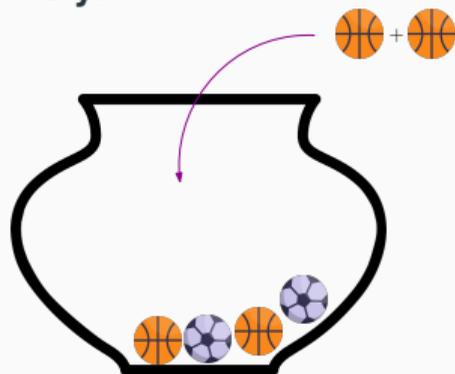
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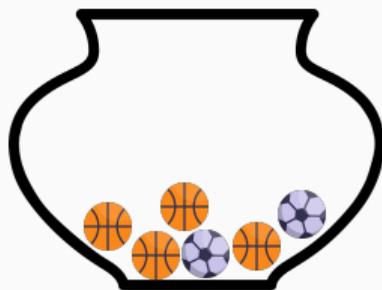
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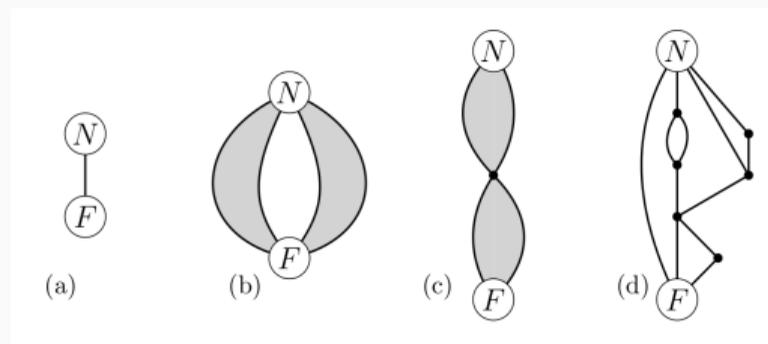
Asymptotic behavior:

Almost surely,

$$\frac{\#\text{orange}}{n} \xrightarrow[n \rightarrow \infty]{} U \sim \mathcal{U}([0, 1])$$

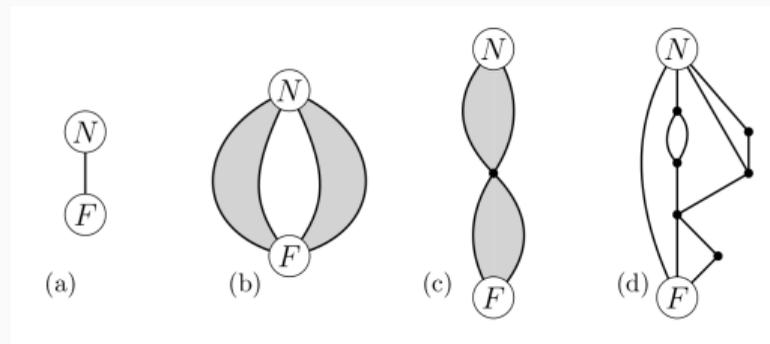
Loop-erased (LE) model on series-parallel graphs

Recursive definition of series-parallel (SP) graphs:



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Recursive definition of series-parallel (SP) graphs:



Theorem (Kious, Mailler, Schapira 22)

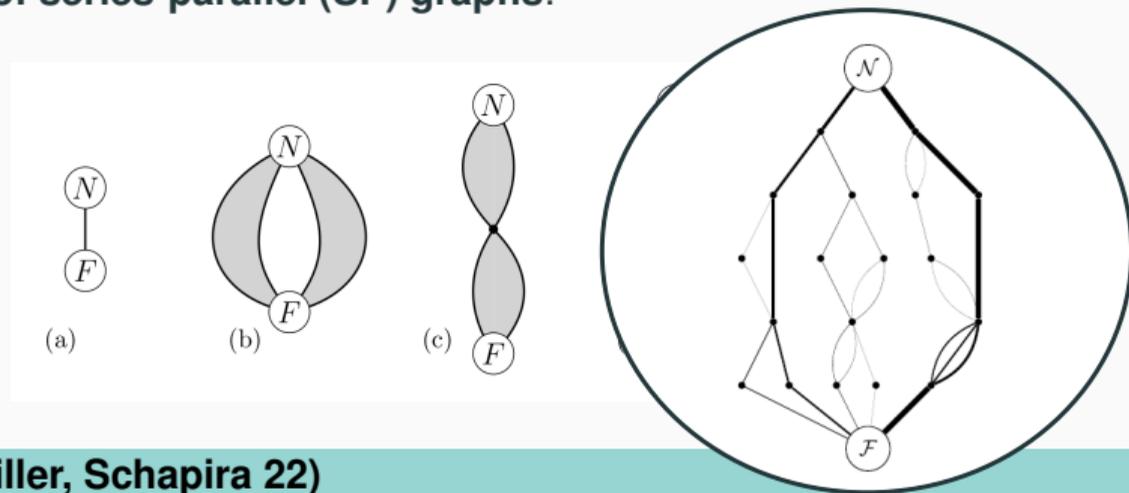
If G is a SP graph, then almost surely,

$$\frac{W_e(n)}{n} \xrightarrow[n \rightarrow \infty]{} \chi_e, \quad \forall e \in E$$

where $(\chi_e)_{e \in E}$ is a random vector such that $\forall e, \chi_e \neq 0 \iff e \in \text{Geodesic}(G)$.

Loop-erased (LE) model on series-parallel graphs

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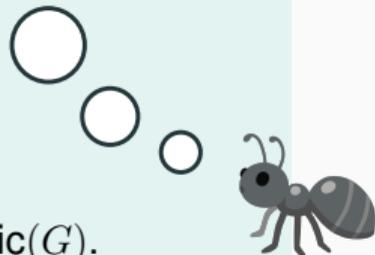


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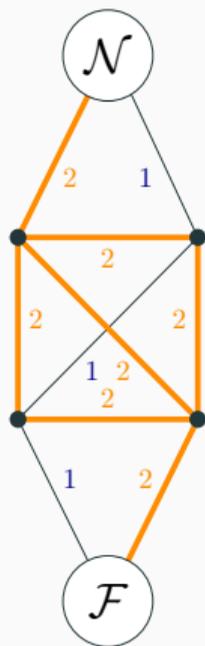
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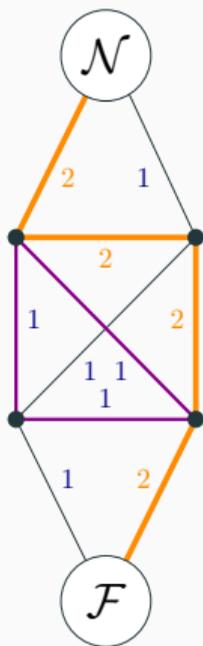


Other reinforcement models



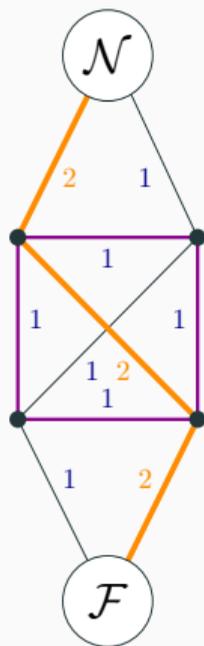
(T) trace:

$$\gamma = X$$



(LE) loop-erased:

$$\gamma = LE(X)$$



(G) geodesic:

$$\gamma = \text{ShortestPath}(X)$$

At each step n :

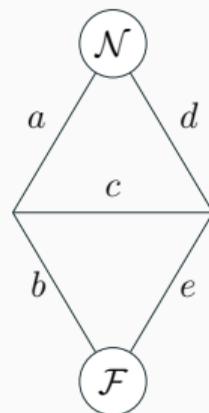
• **random walk** X :

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Geodesic (G) model on the lozenge graph

The lozenge graph:



Theorem (Kious, Mailler, Schapira 22)

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$$\frac{W_i(n)}{n} \xrightarrow[n \rightarrow \infty]{} \chi_i, \quad \forall i \in \{a, b, c, d, e\}$$

where $(\chi_i)_{i \in \{a, b, c, d, e\}}$ is a random vector, such that almost surely, $\chi_c = 0$ and $\chi_a = \chi_b = 1 - \chi_d = 1 - \chi_e \in (0, 1)$.

Conjecture for the loop-erased (LE) and geodesic (G) models

Conjecture (KMS22)

Almost surely,

$$\frac{W_e(n)}{n} \xrightarrow[n \rightarrow \infty]{} \chi_e, \quad \forall e \in E$$

where $(\chi_e)_{e \in E}$ is a random vector such that

(LE) model $\chi_e \neq 0$ a.s. **if and only if** e belongs to a shortest path from N to F

(G) model $\chi_e \neq 0$ a.s. **only if** e belongs to a shortest path from N to F

Conjecture for the loop-erased (LE) and geodesic (G) models

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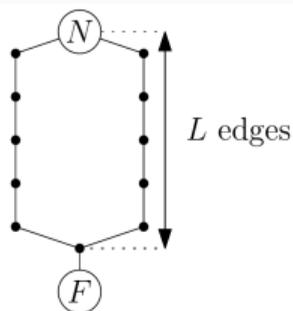
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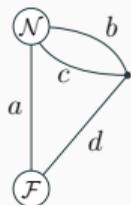


For L large enough, there exists e such that

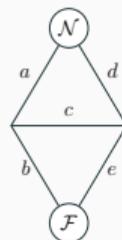
$$\mathbb{P}(W_e(n)/n \rightarrow 0) > 0$$

Trace (T) model (Results from KMS22)

Some examples: the cone and the lozenge



$$\frac{W(n)}{n} \xrightarrow[n \rightarrow \infty]{} (1, 1/3, 1/3, 0)$$

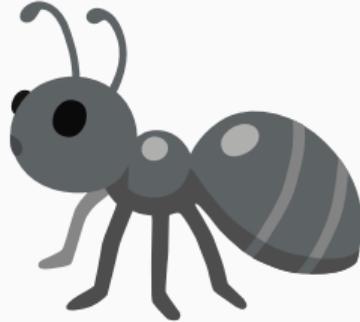


$$\frac{W(n)}{n} \xrightarrow[n \rightarrow \infty]{} (w^*, 1/2, 1/2, w^*, 1/2)$$

Theorem: convergence for a family of *tree-like* graphs (s.t. $G \setminus \{\mathcal{F}\}$ is a tree).

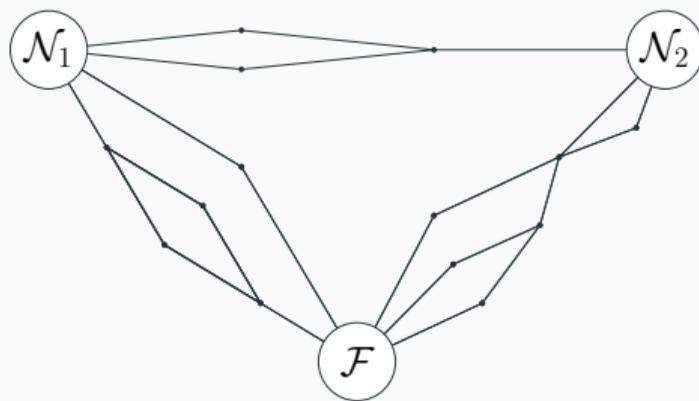
Conjecture: deterministic limit for any graph without multiple-edges adjacent to \mathcal{F} .

The two-nest model



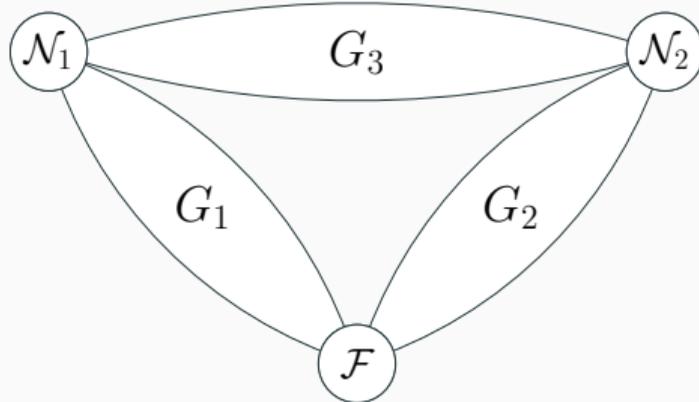
Back to the (LE) model: multinest version

2-nest version: at every step n , $\mathcal{N}(n) = \begin{cases} \mathcal{N}_1 & \text{with proba } \alpha \in (0, 1) \\ \mathcal{N}_2 & \text{with proba } 1 - \alpha \end{cases}$.



Back to the (LE) model: multinest version on triangle-SP graphs

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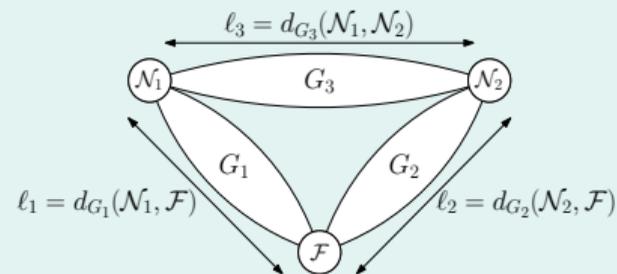


Triangle-SP graph: G_1, G_2, G_3 series-parallel graphs

Our main result: convergence of the 2-nest LE model on triangle-SP graphs

$N_i(n)$ = number of steps at which edges in G_i have been reinforced

Triangle-SP graph



Remark: $\forall n, N_1(n) + N_2(n) = n$.

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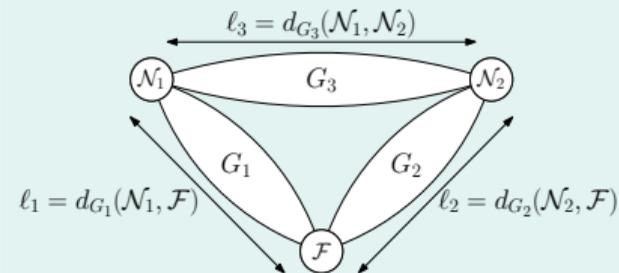
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Theorem (Mailler, V. 25+)

Almost surely,

$$\left(\frac{N_1(n)}{n}, \frac{N_2(n)}{n}, \frac{N_3(n)}{n} \right) \xrightarrow{n \rightarrow \infty} w$$

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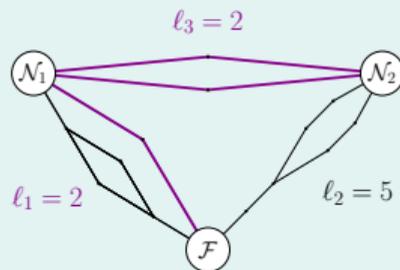
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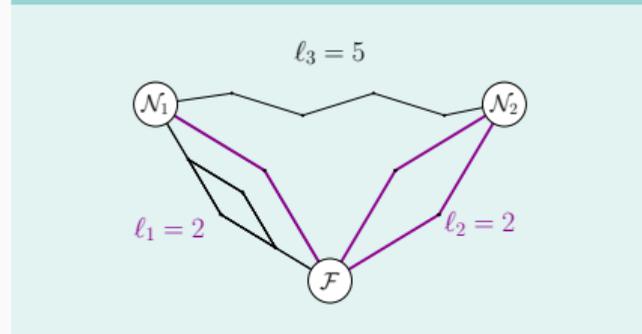
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$$\beta_1 = \frac{\alpha_1 l_1 (l_3 + l_2 - l_1)}{l_1 l_3 + (l_2 - l_1) ((1 - \alpha_1)(l_3 - l_2) + \alpha_1 l_1)}$$

$$\beta_3 = \frac{\alpha_1 l_3 (1 - \alpha_1) (l_1 + l_2 - l_3)}{(l_2 - l_1) (l_1 + l_2 - l_3) \alpha_1 + l_2 (l_1 - l_2 + l_3)}$$

$= d_{G_2}(\mathcal{N}_2, \mathcal{F})$

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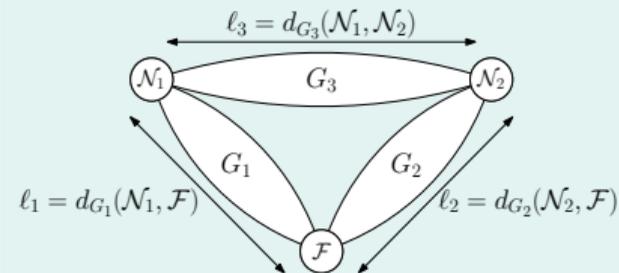
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$$\text{where } \xi_e \neq 0 \iff \begin{cases} e \in \text{Geodesic}(G_i) \\ \lim N_i(n)/n > 0 \end{cases}$$

Triangle-SP graph



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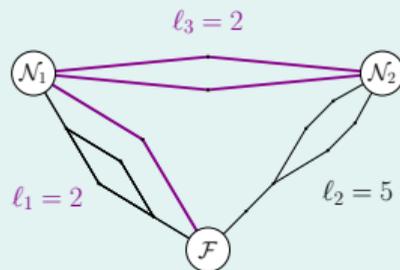
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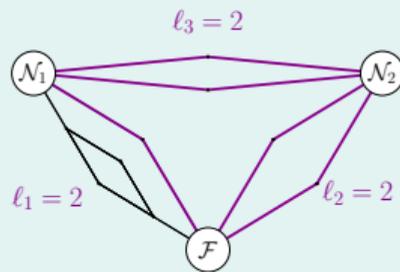
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Triangle-SP graph



When $l_1 = l_2 = l_3$, $\beta_1 = \alpha$ and $\beta_3 = \alpha(1 - \alpha)$.

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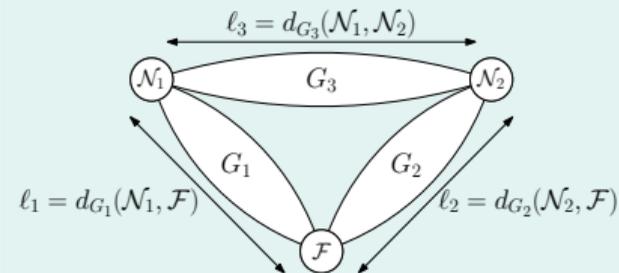
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Toolbox:

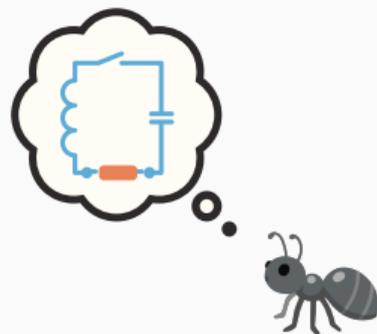
- Pólya urns
- Stochastic approximation theory
- Conductance method and results on the one-nest model on SP-graphs

Toolbox & Proof





Conductance method





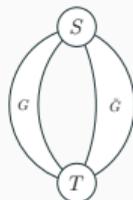
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 See *Probability on trees and networks* (Lyons, Peres 16)

Effective conductance between two vertices - **recursive definition for SP graphs**:



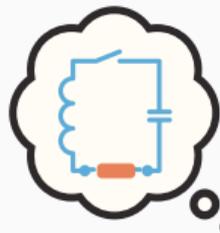
(a) $C_G = w$



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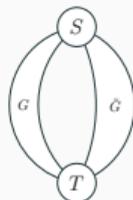
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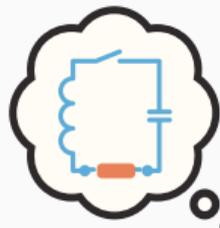
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💡 **Key idea:** the probability that a random walk starting from S hits T before \tilde{T} is $\frac{C_G}{C_G + C_{\tilde{G}}}$.





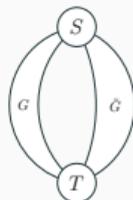
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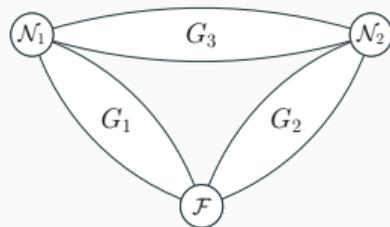
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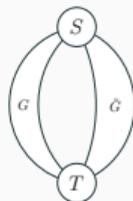
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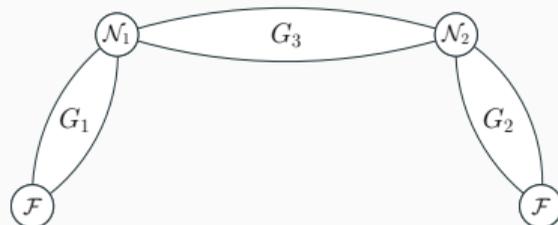
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Asymptotics for the conductances

On SP graphs:



$$h_{\min}(G) = d_G(\mathcal{N}, \mathcal{F})$$

Theorem (Kious, Mailler, Schapira 22):

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- conditional on $\gamma \in G_1$, γ is distributed as γ_1 obtained by doing a (LE) step in G_1 only
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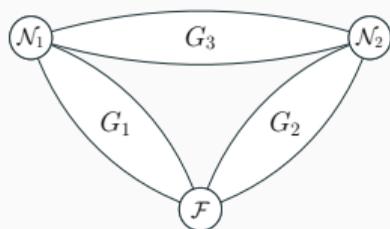
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Two-nest model:



Proposition (Mailler, V. 25+)

For every $i \in \{1, 2, 3\}$,

$$\frac{C_{G_i}(n)}{N_i(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{h_{\min}(G_i)} = \frac{1}{\ell_i}$$

(Partial) summary of the proof

Step 1: Conductance method. $\mathbb{P}(N_1(n+1) - N_1(n) = 1) = \text{Function}(C_{G_1}, C_{G_2}, C_{G_3})$.

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Step 3: Stochastic approximation. The error made in Step 2 is asymptotically negligible.
Thus $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n \geq 0}$ is a stochastic approximation + converges.



Stochastic approximation

📖 See *Random processes with reinforcement* [Pem07]

A process $(X_n)_{n \geq 0}$ is a **stochastic approximation** if

$$X_{n+1} = X_n + \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \forall n$$



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Claim: the process $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n} \right)_{n \geq 0}$ is a stochastic approximation !

Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n} \right)$ and $I = (\mathbb{1}_{N_i(n+1)=N_i(n)+1})_{i=1,3}$

$$\begin{aligned} \frac{N(n+1)}{n+1} &= \frac{N(n) + I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1} \left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n} \right) \\ &= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1} \end{aligned}$$

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Proof that the process is a stochastic approximation

We let, $\forall n$, $N(n) = (N_1(n), N_3(n))$, $\hat{N}(n) = \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n} \right)$ and $I = (\mathbb{1}_{N_i(n+1)=N_i(n)+1})_{i=1,3}$

$$\begin{aligned} \frac{N(n+1)}{n+1} &= \frac{N(n) + I}{n+1} = \frac{N(n)}{n} + \frac{1}{n+1} \left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - \frac{N(n)}{n} \right) \\ &= \frac{N(n)}{n} + \frac{1}{n+1} \left(I - \mathbb{E}[I|\hat{N}(n)] + \mathbb{E}[I|\hat{N}(n)] - p(\hat{N}(n)) + p(\hat{N}(n)) - \frac{N(n)}{n} \right) \\ &= \frac{N(n)}{n} + \frac{F(\hat{N}(n)) + \xi_{n+1} + r_n}{n+1} \end{aligned}$$

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And $\sum_n \frac{\|r_n\|}{n} < \infty$, because $\forall i \in \{1, 2, 3\}, \forall n, N_i(n) \geq n^{\varepsilon_i}$. (*)

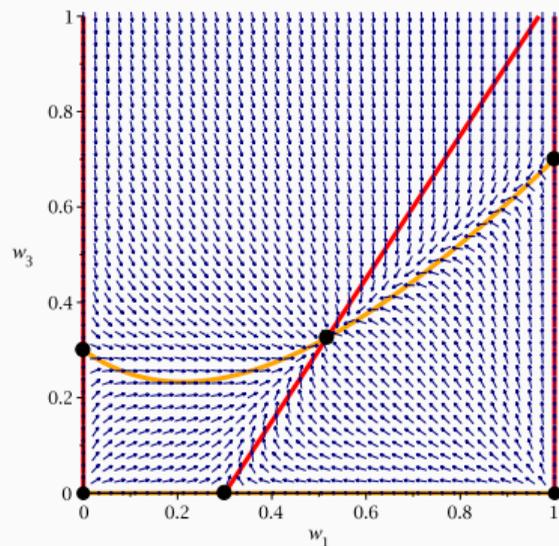
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The ODE method

A process $(X_n)_{n \geq 0}$ is a **stochastic approximation** if

$$X_{n+1} - X_n = \frac{F(X_n) + \xi_{n+1} + r_n}{n+1}, \forall n$$

Vector field $F(w_1, w_3)$:



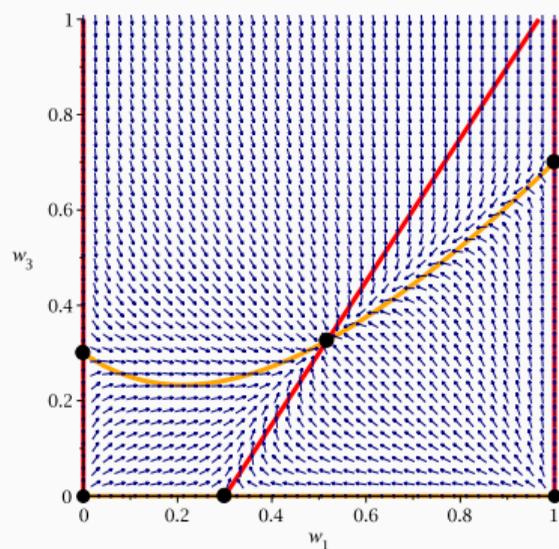
(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

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(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

💡 **Main idea:** if ξ_{n+1} and r_n behave nicely, $(X_n)_{n \leq 0}$ follows the flow of the ODE $\dot{y} = F(y)$!

ODE method

If any solution of the ODE $\dot{y} = F(y)$ converges, then almost surely,

$$\exists \bullet : X_n \xrightarrow{n \rightarrow \infty} \bullet$$

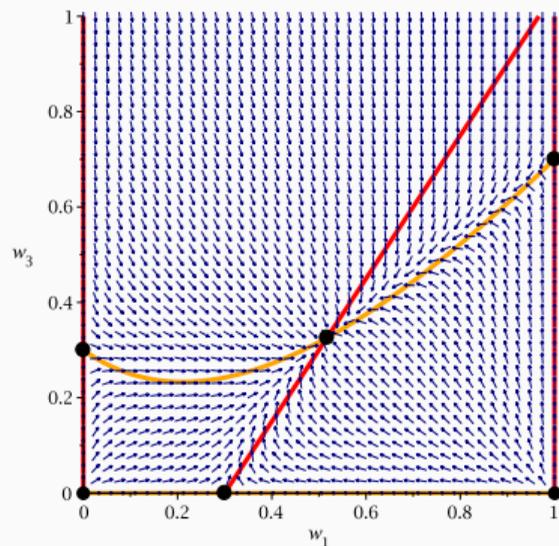
where \bullet is a (random) 0 of F .

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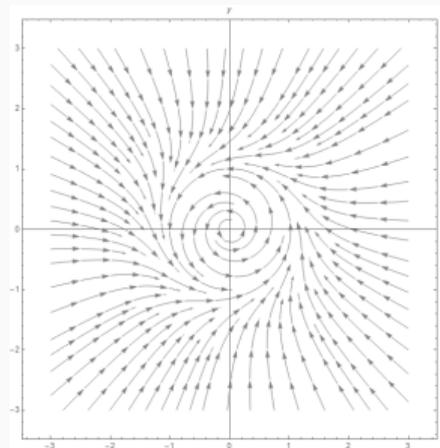
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Vector field $F(w_1, w_3)$:



(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

What does not happen:



Conclusion: any solution to $\dot{y} = F(y)$ starting in $[0, 1]^2$ converges
 $\rightarrow \left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n} \right)$ converges a.s. !

Conclusion of the proof

Step 1: Conductance method. $\mathbb{P}(N_1(n+1) - N_1(n) = 1) = \text{Function}(C_{G_1}, C_{G_2}, C_{G_3})$.

Step 2: $C_{G_i}(n) \underset{n \rightarrow \infty}{\sim} \frac{N_i(n)}{\ell_i}$. (KMS22 + combinatorial analysis of the walks)

Step 3: Stochastic approximation. The error made in Step 2 is asymptotically negligible.
Thus $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right)_{n \geq 0}$ is a stochastic approximation + converges.

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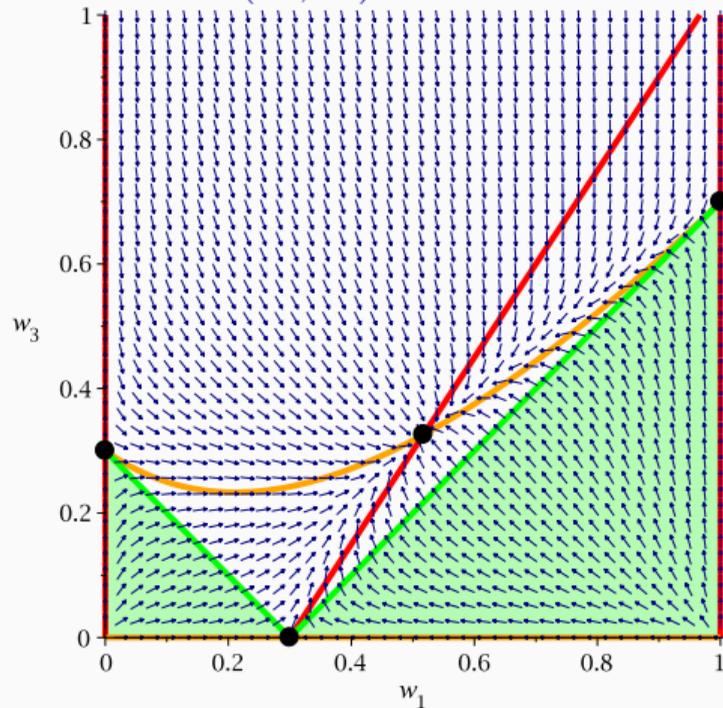
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Step 4: Comparison with generalized Pólya urns to determine the limit.

Eliminating the “bad” zeros

Vector field: $F(w_1, w_3)$



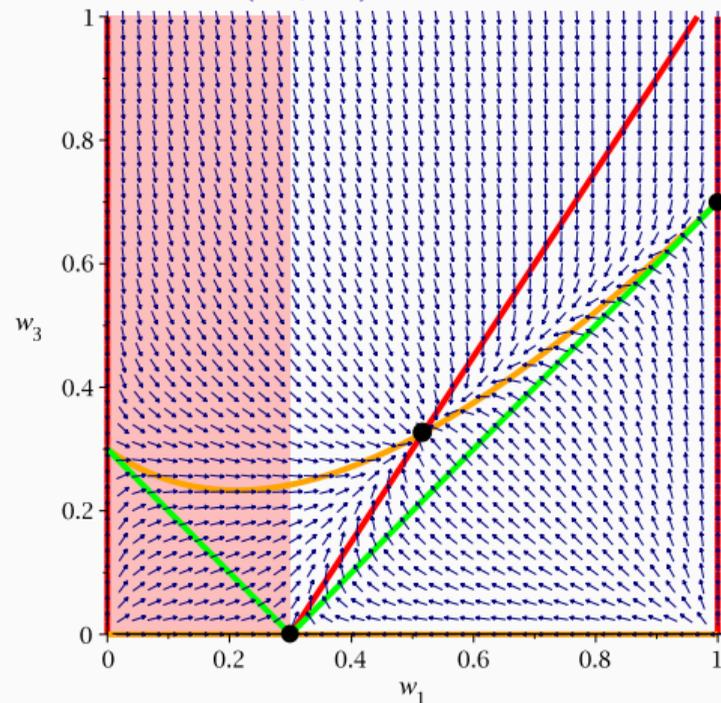
(example with $\ell_1 = 2$, $\ell_2 = 4$ and $\ell_3 = 3$)

$$\liminf_{n \rightarrow \infty} \frac{N_1(n) + N_3(n)}{n} \geq \alpha \text{ and}$$
$$\liminf_{n \rightarrow \infty} \frac{N_2(n) + N_3(n)}{n} \geq 1 - \alpha$$

Lemma

Eliminating the “bad” zeros

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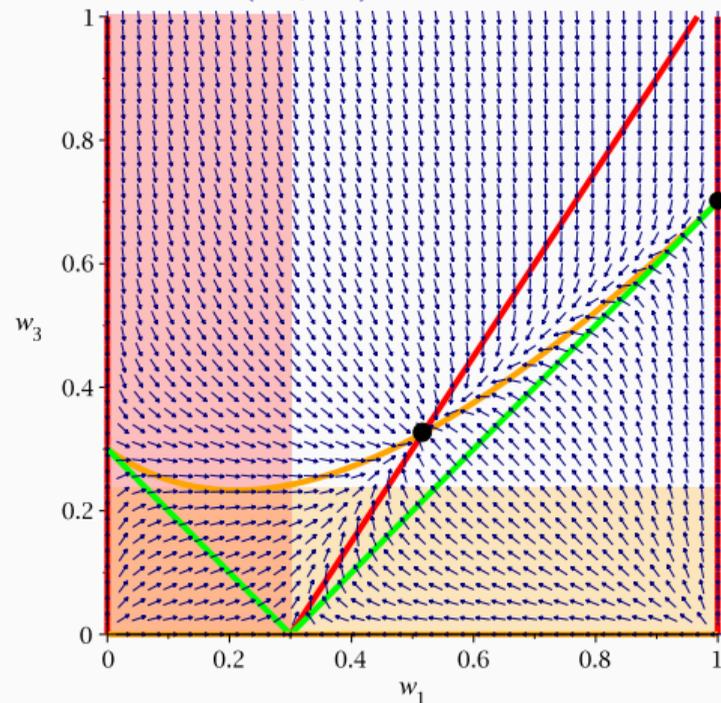
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- $\liminf_{n \rightarrow \infty} \frac{N_1(n)}{n} \geq \alpha$

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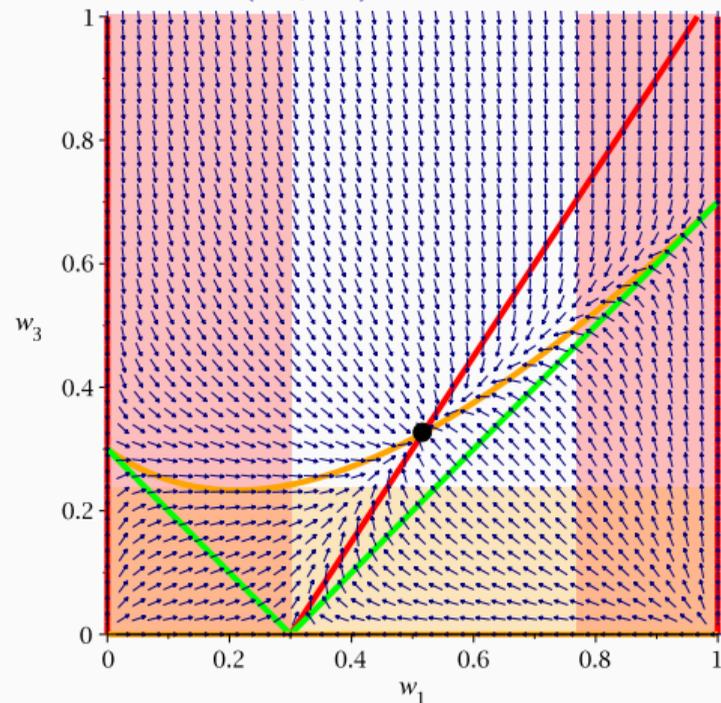
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- if $\ell_3 < \ell_1 + \ell_2$, $\exists c > 0$:

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$$\limsup_{n \rightarrow \infty} \frac{N_1(n)}{n} \leq c'$$



Urn models

$N(n) := \# \text{orange ball}$ at step n . In a classical Pólya urn:

$$\mathbb{P} \left(N(n+1) = N(n) + 1 \mid \frac{N(n)}{n} = w \right) = w$$

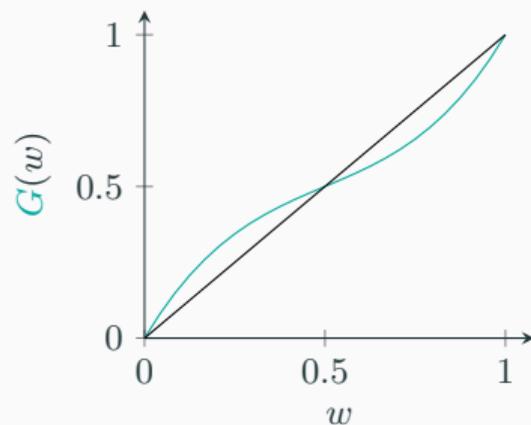




Urn models

$N(n) := \#$  at step n . In a G -urn:

$$\mathbb{P}\left(N(n+1) = N(n) + 1 \mid \frac{N(n)}{n} = w\right) = G(w)$$





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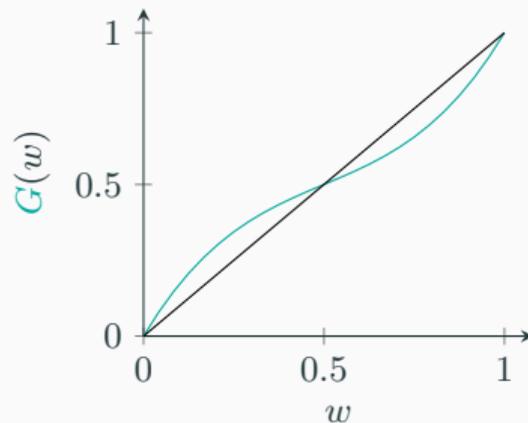
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w is a **stable fixed point** if $G(w) = w$
and $G'(w) \leq 1$

Convergence of G -urn processes

Almost surely, $\frac{N(n)}{n} \xrightarrow[n \rightarrow \infty]{} W$, where
 W is a (random) stable fixed point of
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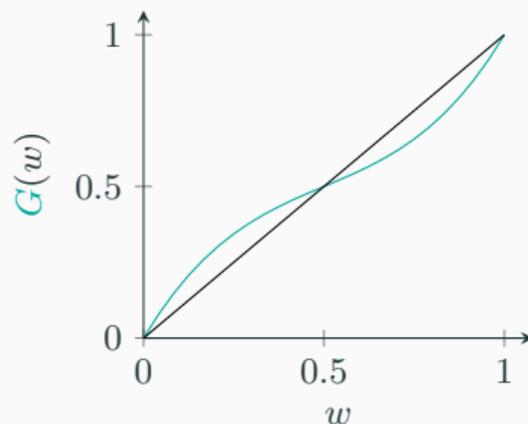
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Examples:

- if $G(w) = w$, $W \sim \mathcal{U}([0, 1])$
- if $G(w) = 2w^3 - 3w^2 + 2w$,
 $W = 0.5$ a.s.





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Use this on our two-dimensional process $(N_1(n), N_3(n))$

If, for any $w_3 \in [0, 1]$,

$$\underbrace{\mathbb{P} \left(N_1(n+1) = N_1(n) + 1 \mid \frac{N_1(n)}{n} = w_1, \frac{N_3(n)}{n} = w_3 \right)}_{\sim F(w_1, w_3)} \geq G(w_1)$$

and if every stable fixed point of G is larger than some c , then

$$\liminf_{n \rightarrow \infty} \frac{N_1(n)}{n} \geq c$$

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Conclusion in the different cases

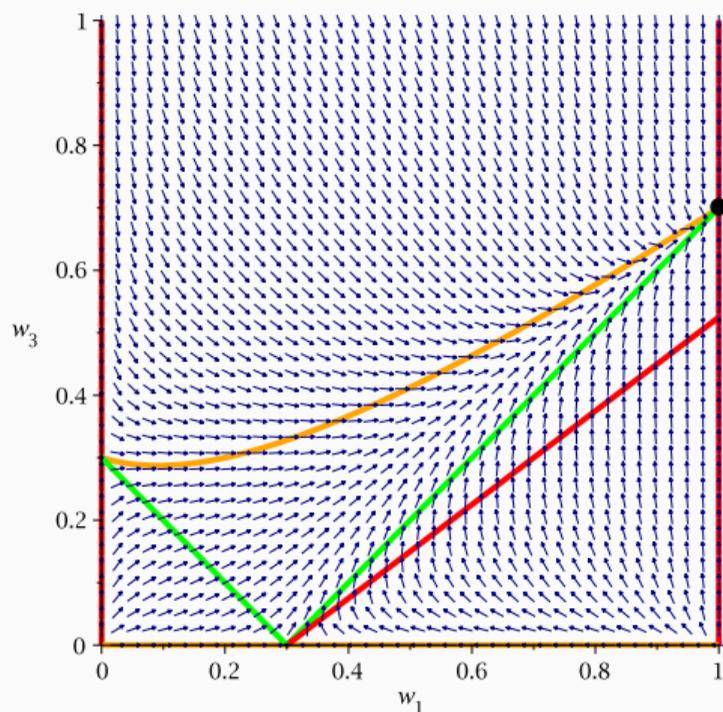


Figure 3: $l_1 = 2$, $l_2 = 6$ and $l_3 = 3$ (case $l_1 + l_3 < l_2$).
 $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \rightarrow (1, \alpha_2)$

Lemma

- $\liminf_{n \rightarrow \infty} \frac{N_1(n)}{n} \geq \alpha_1$
- if $l_3 < l_1 + l_2$, $\exists c > 0$:

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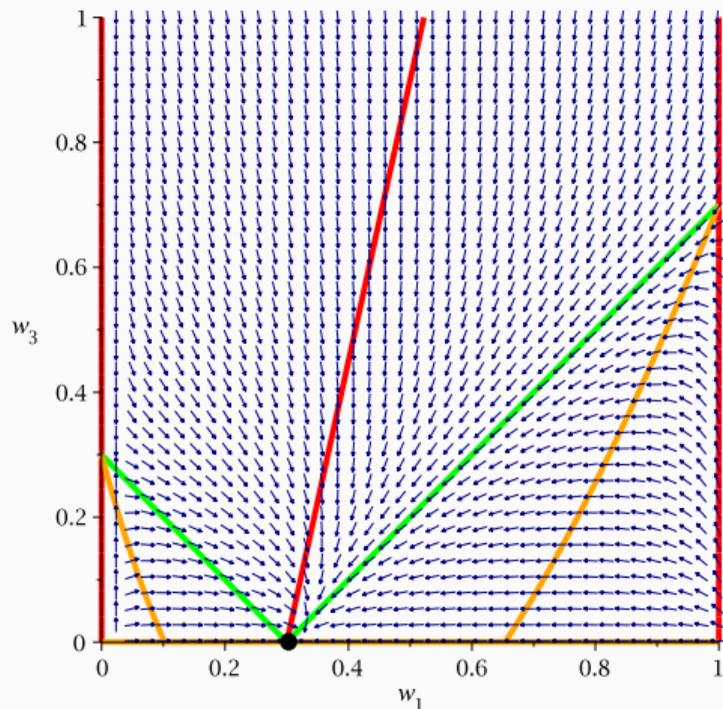


Figure 3: $l_1 = 2$, $l_2 = 4$ and $l_3 = 9$ (case $l_1 + l_2 < l_3$).
 $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \rightarrow (\alpha_1, 0)$

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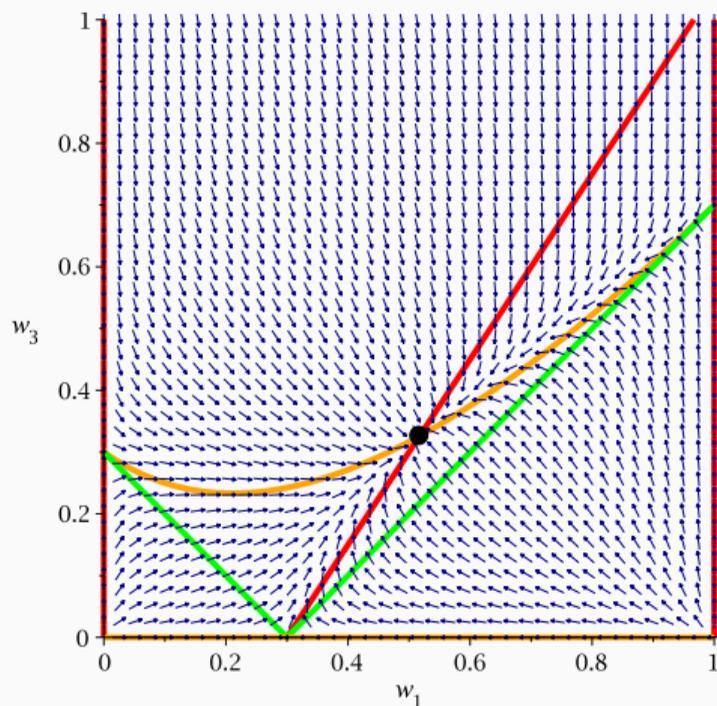


Figure 3: $l_1 = 2, l_2 = 4$ and $l_3 = 3$ (case $l_1 + l_3 \geq l_2$ and $l_1 + l_2 \geq l_3$).
 $\left(\frac{N_1(n)}{n}, \frac{N_3(n)}{n}\right) \rightarrow (\beta_1, \beta_3)$

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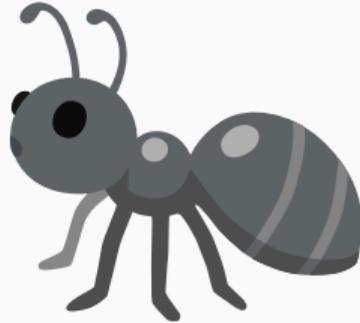
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Step 4: Comparison with generalized Pólya urns to determine the limit.

A lot of unanswered questions remain

- Can we characterize the distribution of the edge weights' limit ?
- What happens on other families of graphs ?
- Or if we consider graphs with more nests ?
- Evaporation of the pheromones ?

Thank you !



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