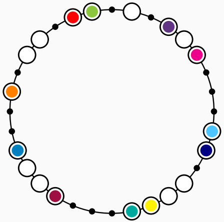
The golf model on $\mathbb{Z}/n\mathbb{Z}$ and on $\mathbb{Z}$

Journée "Systèmes de particules en interaction"

Zoé Varin

May 11th, 2026

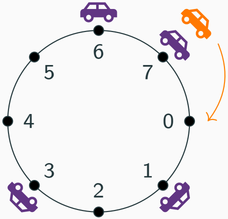


Short presentation

- Since November: Postdoc at IRIF, in the Combinatorics team
  - working with Marie Albenque and Guillaume Chapuy
  - funded by the ComplexCité consortium
- PhD in Bordeaux (LaBRI) supervised by Jean-François Marckert
  - the golf model and other dispersion models (two papers, the second with J.-F. Marckert)
  - the ant model (with Cécile Mailler)

A first interacting particle system:

The parking model



hashing with linear probing $\leftrightarrow$ the parking model (Konheim, Weiss 66; Knuth 73; Flajolet, Poblete, Viola 98)

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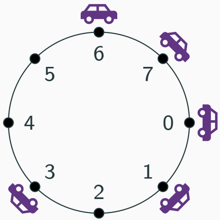

hashing with linear probing ↔ the parking model (Konheim, Weiss 66; Knuth 73; Flajolet, Poblete, Viola 98)

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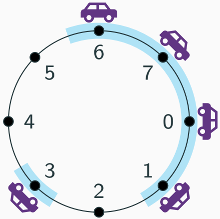
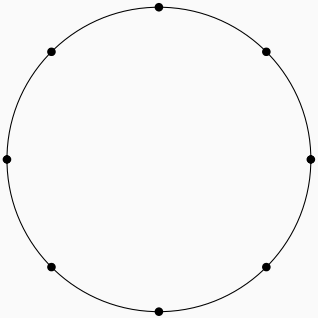


hashing with linear probing $\leftrightarrow$ the parking model (Konheim, Weiss 66; Knuth 73; Flajolet, Poblete, Viola 98)

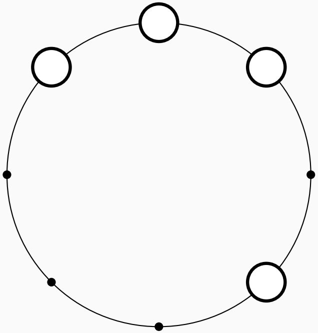
complexity of inserting a new data $\leftrightarrow$ size of the largest block $+1$

Definition of the golf model



$G = (V, E)$, here $\mathbb{Z}/n\mathbb{Z}$

- **A (random) initial configuration**



N_h holes:

$$\mathrm{H}^{init} = \{\bigcirc\}$$

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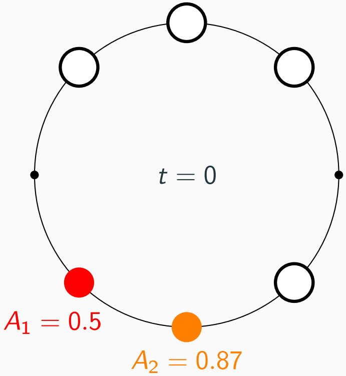

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One activation clock per ball:

$$\boldsymbol{A}_v \sim \mathcal{U}([0, 1])$$

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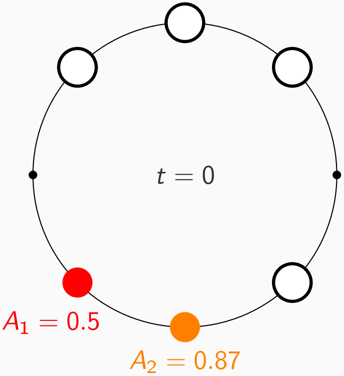

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The golf model on $\mathbb{Z}/n\mathbb{Z}$: 

$$(\boldsymbol{B}^{init}, \mathrm{H}^{init}) \sim \mathrm{Unif}\left(\binom{\mathbb{Z}/n\mathbb{Z}}{N_b, N_h, n - N_h - N_b}\right)$$

N_h and N_b fixed, with $N_h \geq N_b$

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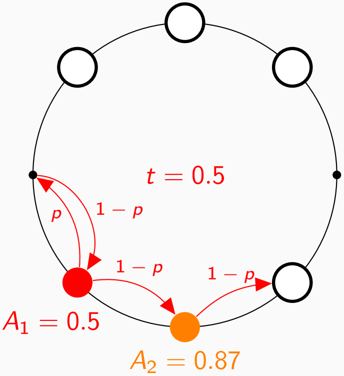
- **Random dynamics**



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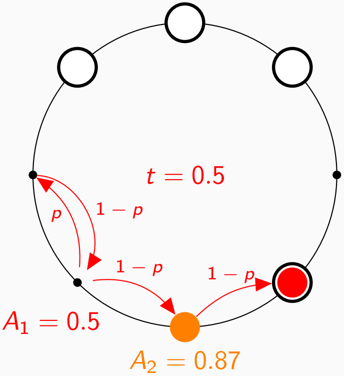

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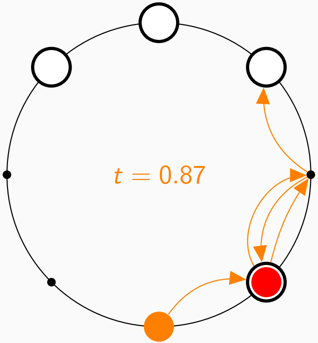

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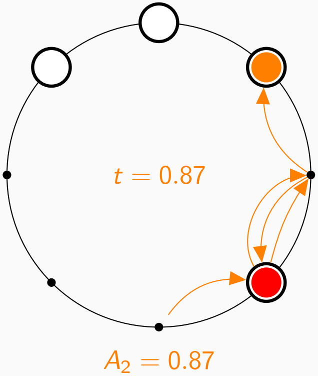

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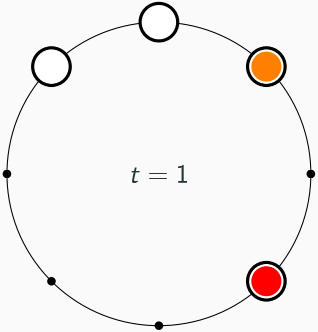

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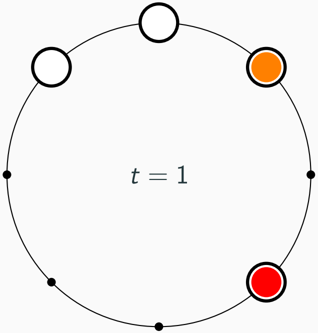

Free holes:

$$H^{L} = \left\{ \text{positions of } \bigcirc \text{ at } t = 1 \right\}$$

Occupied holes:

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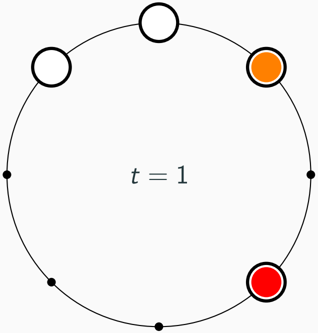

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The random variable $\mathrm{H}^{\boldsymbol{L}}$ is well-defined.

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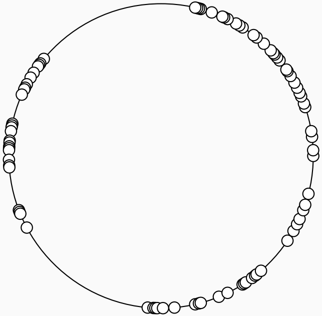

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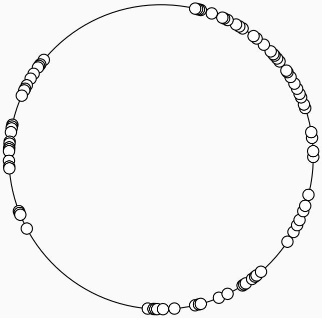

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Commutation property (Diaconis-Fulton 91)

The distribution of $\mathrm{H}^{\boldsymbol{L}}$ and $\mathrm{H}^{\boldsymbol{o}}$ do not depend on the activation order of the balls

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Key tool for the proof: changing point of view - heaps of arrows

- on every vertex: a heap of arrows

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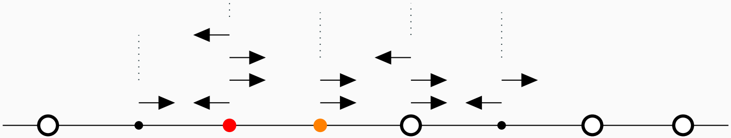

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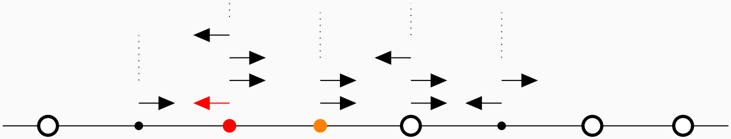

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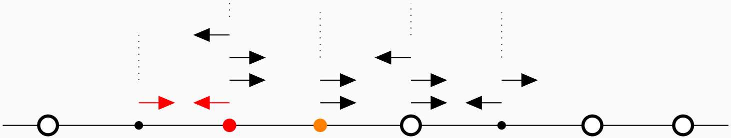

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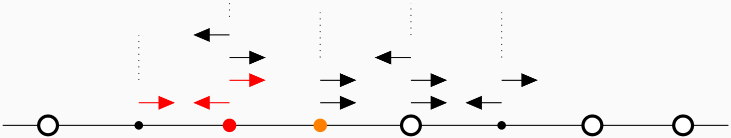

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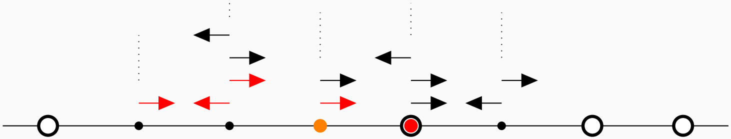

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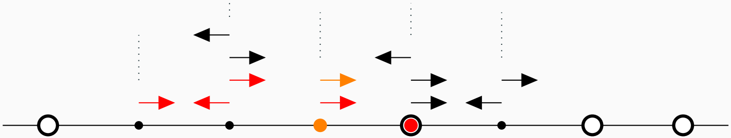

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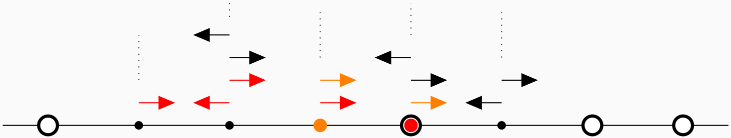

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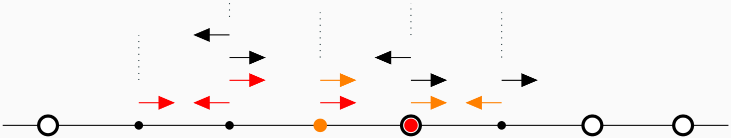

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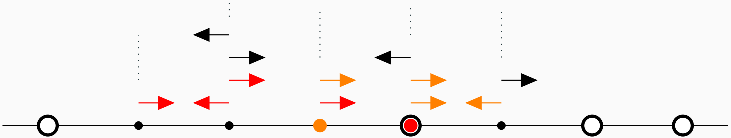

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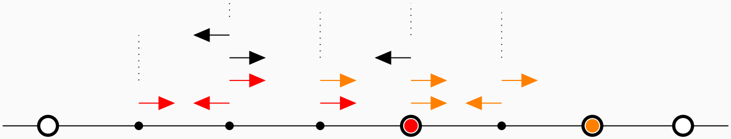

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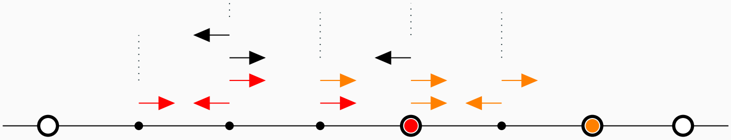

- commutation: same arrows used and same holes filled

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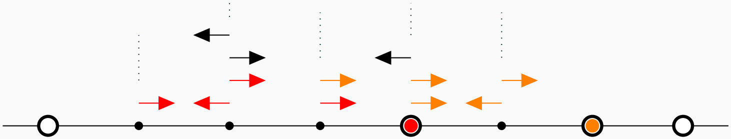

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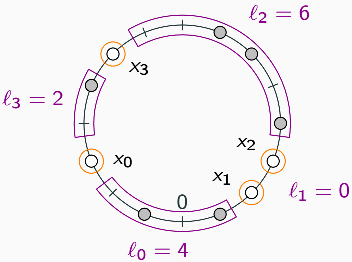
This commutation property is valid for a wide family of interacting particle systems !

Theorem (Universality of $\mathrm{H}^{\boldsymbol{L}}$ - V. 25)

For every p,

$$\mathbb{P}^{n,N_{\mathbf{b}},N_{\mathbf{h}},p}\left(\mathrm{H}^{\boldsymbol{L}} = X\right) = \frac{1}{\binom{n}{N_{\mathbf{b}},N_{\mathbf{h}},n-N_{\mathbf{b}}-N_{\mathbf{h}}}} \sum \prod_{i=0}^{N_{\boldsymbol{L}}-1} \frac{1}{b_i+1}\binom{\ell_i}{b_i, b_i, \ell_i - 2b_i}$$

where the sum is taken over the set of $(b_i)_{0\leq i<N_L}$ such that $\sum_i b_i = N_{\mathrm{b}}$ and $\forall i, 2b_i \leq \ell_i$.



$X = \{x_0 \ldots, x_{N_L-1}\}$, $N_{\boldsymbol{L}} = N_{\mathrm{h}} - N_{\mathrm{b}}$.

$0 < x_1 < \ldots < x_{N_L-1} < x_0 \leq n$

$\forall i,\ \ell_i := (x_{i+1} - x_i - 1) \bmod n$

Distribution of $\mathsf{H}^{\boldsymbol{L}}$ (n fixed)

Theorem (Universality of $\mathsf{H}^{\boldsymbol{L}}$ - V. 25)

For every local and invariant under rotation policy,

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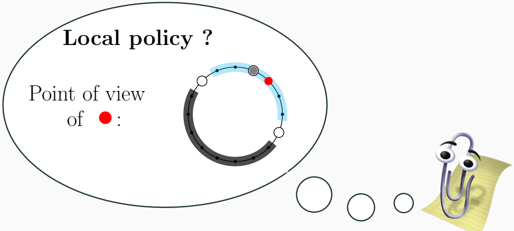

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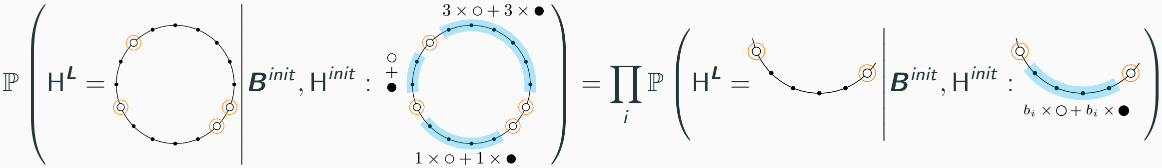
Main idea of the proof:



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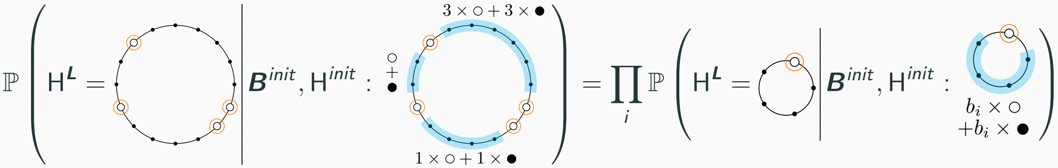
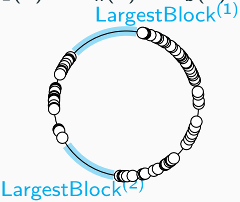


Asymptotic behavior of H^L

No neutral vertices: $n = N_\mathsf{h}(n) + N_\mathsf{b}(n)$. Number of free holes at time $t = 1$: $N_L(n) = N_\mathsf{h}(n) - N_\mathsf{b}(n)$.

LargestBlock$^{(i)}$ = size of the ith largest block of $V \setminus \mathsf{H}^L$.



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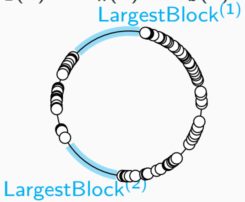


Theorem (linear case - V. 25)

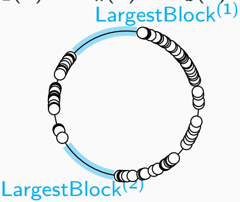
If $N_L = N_L(n) \sim an$, *with* $a > 0$, *then* $\exists \alpha, \beta > 0$ *such that*

$$\mathbb{P}\left(\alpha \leq \frac{\mathrm{LargestBlock}^{(1)}}{\log n} \leq \beta\right) \underset{n\to\infty}{\to} 1$$

Asymptotic behavior of H^L

No neutral vertices: $n = N_h(n) + N_b(n)$. Number of free holes at time $t = 1$: $N_L(n) = N_h(n) - N_b(n)$.

$\text{LargestBlock}^{(i)}$ = size of the ith largest block of $V \backslash \mathsf{H}^L$.



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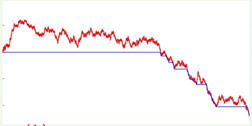
Theorem (phase transition - V. 25)

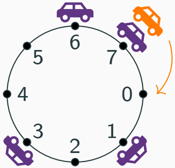
- *If $N_L \ll \sqrt{n}$, then* $\frac{\text{LargestBlock}^{(1)}}{n} \overset{\mathbb{P}}{\to} 1$.
- *If $N_L \gg \sqrt{n}$, then* $\frac{\text{LargestBlock}^{(1)}}{n} \overset{\mathbb{P}}{\to} 0$.
- *Description of the **phase transition**: if $N_L/\sqrt{n} \to \lambda \geq 0$,*
  $\left(\frac{\text{LargestBlock}^{(i)}}{n}, i \geq 1\right) \overset{(d)}{\to} \left(\text{LargestExc}(B^{(\lambda)}), i \geq 1\right)$.



$B^{(\lambda)}$ *such that* $\tau_{-\lambda} = 1$

Back to the parking model

The parking model:



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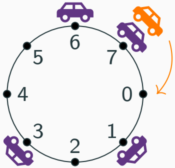

Theorem [Pittel 87, Chassaing-Louchard 02]

- If $N_L = N_L(n) \sim an$, $a > 0$, then $\mathrm{LargestBlock}^{(1)}$ converges in probability:
$$\mathrm{LargestBlock}^{(1)} = \frac{\log n - 3/2 \log\log n}{a - 1 - \log a} + O(1).$$
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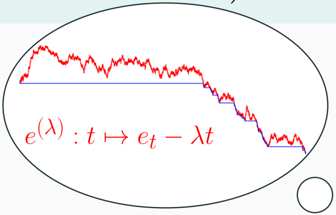

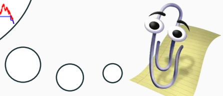

Back to the parking model

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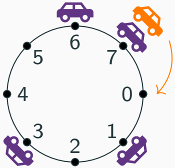

Generalized parking: local and invariant under rotation parking policy (Nadeau 23)

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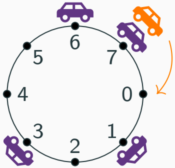
Theorem (V. 25 - Universality of H^L for the generalized parking)

$$\mathbb{P}^{n,N_{\mathbf{b}},N_{\mathbf{h}},p}\left(\mathrm{H}^L = X\right) = \frac{1}{n^{N_{\mathbf{b}}}} \binom{n - N_L}{\ell_1, \ldots, \ell_{N_L}} \prod_{i=1}^{N_L} (\ell_i + 1)^{\ell_i - 1}$$

Back to the parking model

The parking model:



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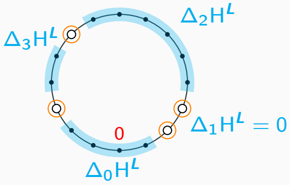
How do we get asymptotic results ?

$\Delta_i \mathrm{H}^{\boldsymbol{L}}$ = size of the ith block of $V \backslash \mathrm{H}^{\boldsymbol{L}}$.



Size of the blocks when $n = N_{\mathrm{h}}(n) + N_{\mathrm{b}}(n)$

$$\mathbb{P}^{n, N_{\mathbf{b}}, N_{\mathbf{h}}, p}\left(\forall i, \Delta_i \mathrm{H}^{\boldsymbol{L}} = 2b_i\right) = \frac{1}{\binom{n}{N_{\mathbf{b}}}}(2b_0 + 1) \prod_{i=0}^{N_{\boldsymbol{L}}-1} C_{b_i}$$

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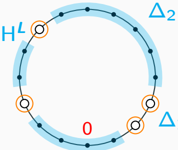

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Some combinatorics...

$\mathrm{Forests}(n, N_{\boldsymbol{L}}) = \{$forests of binary trees with n vertices et $N_{\boldsymbol{L}}$ roots$\}$

$\prod_{i=0}^{N_{\boldsymbol{L}}-1} C_{b_i}$ = number of forests such that the ith tree has b_i internal vertices

$\rightarrow$ study of excursions of $\boldsymbol{p} \sim \mathrm{Unif}\left(\mathrm{Paths}(n, N_{\boldsymbol{L}})\right)$ (where $\mathrm{Paths}(n, N_{\boldsymbol{L}}) = \{$paths such that $\tau_{-N_{\boldsymbol{L}}} = n\}$)

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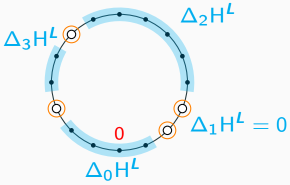

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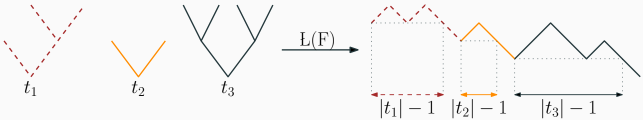

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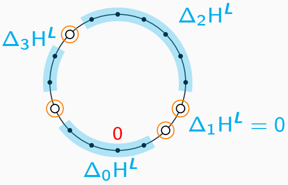

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...and some probabilities: if $\boldsymbol{p} \sim \mathcal{U}\left(\mathrm{Paths}(n, N_{\boldsymbol{L}})\right)$, then

$$\left(\frac{\boldsymbol{p}(2nt)}{\sqrt{2n}}\right)_{t\in[0,1]} \xrightarrow{(d)} B^{(\lambda)},$$

in $(C([0,1],\mathbb{R}), ||.||_\infty)$, where $B^{(\lambda)}$ is a brownian motion B conditioned by $\tau_{-\lambda}(B) = 1$.

Then, for every k, the k largest excursion lengths of $\boldsymbol{p}$ converge (up to normalisation) to those of $B^{(\lambda)}$ (Aldous lemma).

The parking and golf models on $\mathbb{Z}$

The parking model on $\mathbb{Z}$

Parking on the integers, Przykucki, Roberts, Scott 23.

Parking on transitive unimodular graphs., Damron, Gravner, Junge, Lyu, Sivakoff 19.

Initial configuration for every vertex u, independently from the other vertices:

- initial state: a ball with proba d_b OR a hole with proba d_h, $0 \leq d_b \leq d_h$

All the balls move simultaneously ! (at each step, +1 with proba p, -1 with proba $1 - p$)

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The golf model on $\mathbb{Z}$ - Definition

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- activation clock: $\boldsymbol{A}_u \sim \mathcal{U}([0,1])$

When activated, the ball does *instantaneously* moves to a free hole (the first hole hit by a random walk of parameter p).

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Theorem (V. 25)

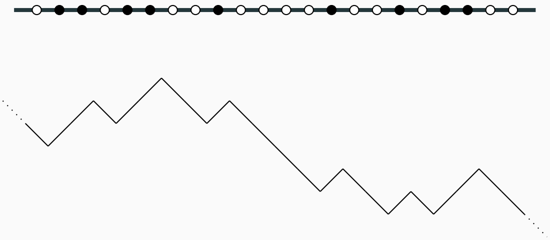
The golf model on $\mathbb{Z}$ is well-defined (even when $d_b = d_h$).

Proof that the golf model on $\mathbb{Z}$ is well-defined

Theorem (V. 2024+)

The golf model on $\mathbb{Z}$ is well-defined.

Key : encoding of the initial configuration



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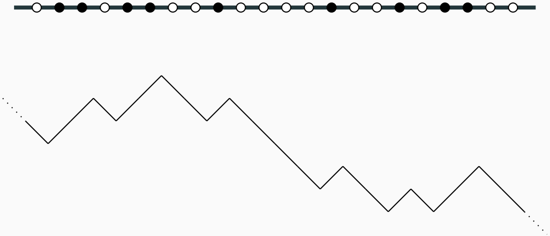

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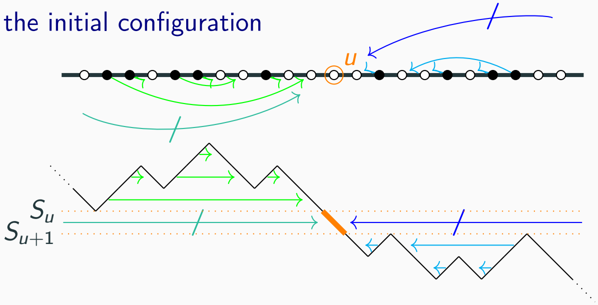

(orange edge) sees no other edge $\implies$ u is never filled by a ball

Proposition (cas $d_b < d_h$)

Almost surely, there exists an infinite number of (orange edges) .

The golf model on $\mathbb{Z}$ - Distribution of $\mathsf{H}^{\boldsymbol{L}}$

$\left(\Delta_i \mathsf{H}^{\boldsymbol{L}}\right)_{i\in\mathbb{Z}}$ = block sizes process. Assuming that $d_{\mathrm{b}} + d_{\mathrm{h}} = 1$,

Theorem (V. 25)

- *If $d_{\mathrm{b}} < d_{\mathrm{h}}$: there exists $\mathcal{G}, \mathcal{H}$ and λ (explicit) such that, for all $R > 0$,*

$$\mathbb{P}\left(\Delta_i \mathsf{H}^{\boldsymbol{L}} = 2b_i, -R \leq i \leq R\right) = \frac{(2b_0+1)\lambda^{2b_0} C_{b_0}}{\mathcal{H}(\lambda)} \prod_{i=-R, i\neq 0}^{R} \frac{\lambda^{2b_i} C_{b_i}}{\mathcal{G}(\lambda)}$$

- *If $d_{\mathrm{b}} = d_{\mathrm{h}}$, then almost surely, $\mathsf{H}^{\boldsymbol{L}} = \emptyset$.*

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$$= \lim_{n\to\infty} \mathbb{P}^n \left(\Delta_i \mathrm{H}^{\boldsymbol{L}(n)} = 2b_i, -R \leq i \leq R\right)$$

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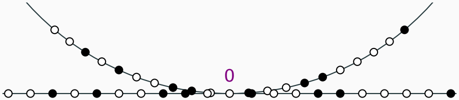
Key: coupling with the golf model on $\mathbb{Z}/n\mathbb{Z}$



$\frac{N_\mathbf{b}(n)}{n} \to d_\mathrm{b}$, $\frac{N_\mathbf{h}(n)}{n} \to d_\mathrm{h}$

local environment: similar + suffices

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Same final configuration as the parking !

Cost of generalized parking models (with Jean-François Marckert)

(or jump to conclusion)

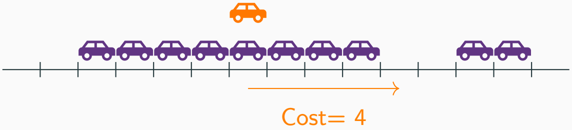
Cost of (generalized) parking models

$$\text{GlobalCost}_n(r) = \sum_{i=1}^{r} \text{cost to park the } i\text{th car}$$
$$= \sum_{i=1}^{r} \text{Cost}_{t_i}^{i}, \text{ where } t_i \text{ size of the block where the } i\text{th car falls}$$

Two important objects

- the list t_i
- $\mathbb{E}[\text{Cost}_\ell^{\cdot}]$ and $\text{Var}(\text{Cost}_\ell^{\cdot})$

Cost of the parking: $\text{Cost}_\ell^{\cdot} \sim \text{Unif}(\llbracket 1, \ell \rrbracket)$



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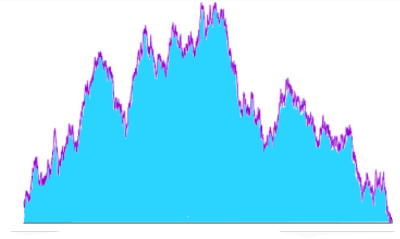
If $r = n$, (Flajolet-Poblete-Viola 98)

$$\frac{\text{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e_t\,dt$$

Illustration of $\int_0^1 e_t\,dt$:



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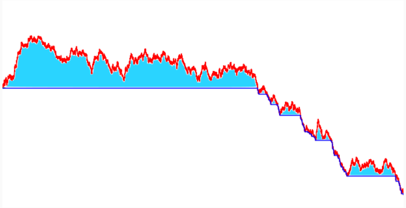
$$\frac{\text{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e_t dt$$

If $r = \lfloor n - \lambda\sqrt{n}\rfloor$, (Chassaing-Louchard 02)

$$\frac{\text{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e^{(\lambda)}(t) - \inf_{s\le t} e^{(\lambda)}(s)dt =: F(\lambda)$$

Illustration of $\int_0^1 e^{(\lambda)}(t) - \inf_{s\le t} e^{(\lambda)}(s)dt$:



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Two important objects

- the list t_i
- $\mathbb{E}\left[\text{Cost}_\ell^{\cdot}\right]$ and $\text{Var}\left(\text{Cost}_\ell^{\cdot}\right)$

Cost of the parking: $\text{Cost}_\ell^{\cdot} \sim \text{Unif}\left(\llbracket 1, \ell \rrbracket\right)$

If $r = n$, (Flajolet-Poblete-Viola 98)

$$\frac{\text{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e_t \, dt$$

If $r = \lfloor n - \lambda\sqrt{n} \rfloor$, (Chassaing-Louchard 02)

$$\frac{\text{GlobalCost}_n(r)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \int_0^1 e^{(\lambda)}(t) - \inf_{s\le t} e^{(\lambda)}(s) dt =: F(\lambda)$$

Theorem (Marckert-V.25+)

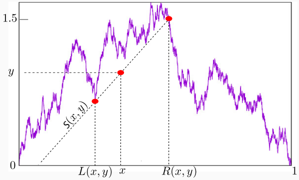
Convergence in distribution of

$$\frac{GlobalCost_n\left(\lfloor n - \lambda\sqrt{n} \rfloor\right)}{\sqrt{n}\alpha_n}$$

under some hypotheses on $\mathbb{E}\left[Cost_k\right]$ *and* $Var\left(Cost_k\right)$, *towards an explicit limit*

Cost of (generalized) parking models

$$\mathrm{GlobalCost}_n(r) = \sum_{i=1}^{r} \text{cost to park the } i\text{th car}$$
$$= \sum_{i=1}^{r} \mathrm{Cost}^i_{t_i}, \text{ where } t_i \text{ size of the block where the } i\text{th car falls}$$



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Case of parameter p random walks

If $p \neq 1/2$,

$$\frac{\mathrm{GlobalCost}_n(\lfloor n - \lambda\sqrt{n}\rfloor)}{n^{3/2}} \xrightarrow[n\to\infty]{(d)} \frac{1}{|2p-1|} F(\lambda)$$

If $p = 1/2$,

$$\frac{\mathrm{GlobalCost}_n(\lfloor n - \lambda\sqrt{n}\rfloor)}{n^{5/2}} \xrightarrow[n\to\infty]{(d)} \frac{1}{3} G(\lambda)$$

where $G(\lambda) = \int_0^1 \int_0^{e_x} (R(x,y) - L(x,y)) \mathbb{1}_{S(x,y)\ge\lambda}\,dydx$

Perspectives

Perspectives

- $\rightarrow$ The golf model on the infinite binary tree $\mathbb{B}$
- $\rightarrow$ The golf model on $\mathbb{Z}^2$
- $\rightarrow$ The golf model on random trees (or maps)

Merci !

Perspectives

$\to$ The golf model on the infinite binary tree $\mathbb{B}$

Theorem (Albenque, Chapuy, Varin 26+)

The golf model is well-defined on $\mathbb{B}$ if $d_{\mathrm{b}} \leq 1/64$.

Conjecture

The golf model is well-defined on $\mathbb{B}$ if $d_{\mathrm{b}} \leq 1/2$.

$\to$ The golf model on $\mathbb{Z}^2$

$\to$ The golf model on random trees (or maps)

Merci !

Continuous dispersion models - illustration